Exactness of Lasserre relaxations and when real closed fields help

based on joint work with Tom-Lukas Kriel and with Luis Felipe Vargas

Markus Schweighofer

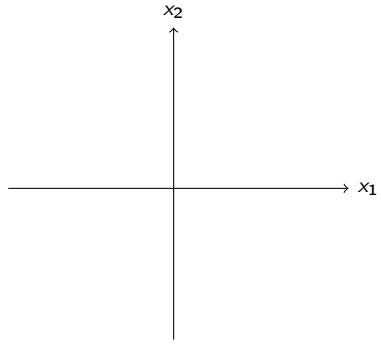
Universität Konstanz

Tensors: geometry, optimization, and applications — TENORS Doctoral Days
Florence, November 25, 2026

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrcrcrcl}
A & 1 & - & x_1^2 & - & x_2^2 & \geq & 0 \\
B & & & & & x_1 x_2 & \geq & 0
\end{array}
$$



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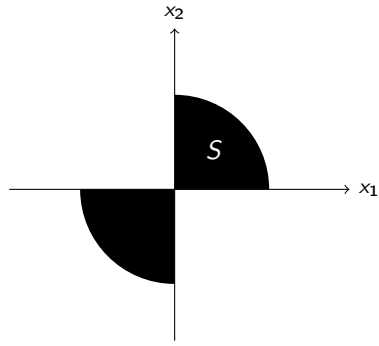
\end{array}
$$

The solution set S is the unit disk in the first and third quadrant.



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\end{array}
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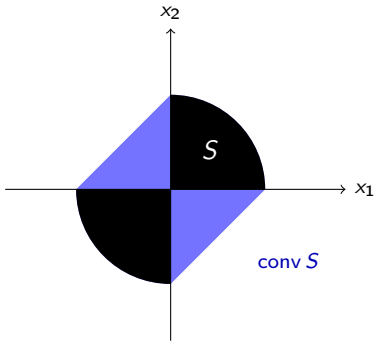
The solution set S is the unit disk in the first and third quadrant.

We would like to optimize linear functions over S, that is, to understand conv S.



System of polynomial inequalities

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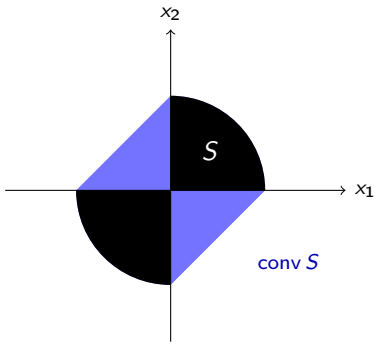
\begin{array}{lrcrcl}
A & 1 & - & x_1^2 & - \quad x_2^2 & \geq 0 \\
B & & & & x_1 x_2 & \geq 0
\end{array}
$$

The solution set S is the unit disk in the first and third quadrant.

We would like to optimize linear functions over S, that is, to understand conv S.

Linearize: replace each nonlinear monomial by a new variable.



System of polynomial inequalities

Very naive linearization

$$\begin{array}{lrcrcrcl} A & 1 & - & y_1 & - & x_2^2 & \geq & 0 \\ B & & & & & x_1 x_2 & \geq & 0 \end{array}$$

The solution set S is the unit disk in the first and third quadrant.

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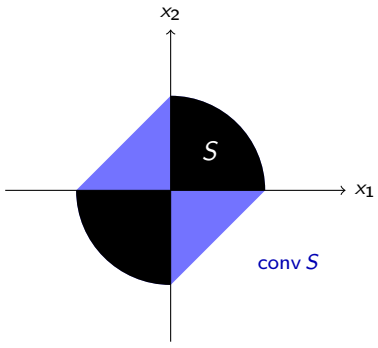
Linearize: replace each nonlinear monomial by a new variable.



System of polynomial inequalities

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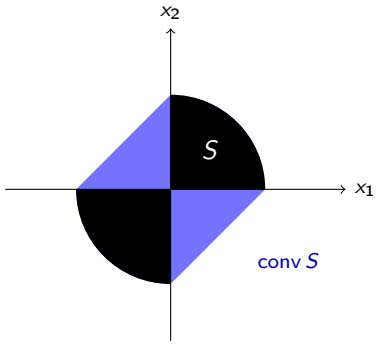
The solution set S is the unit disk in the first and third quadrant.

We would like to optimize linear functions over S, that is, to understand conv S.

Linearize: replace each nonlinear monomial by a new variable.



System of linear inequalities

Very naive linearization

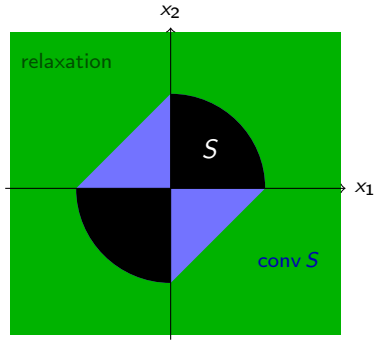
$$\begin{array}{lrcrcrcl} A & & 1 & - & y_1 & - & y_2 & \geq & 0 \\ B & & & & & & y_3 & \geq & 0 \end{array}$$

The solution set S is the unit disk in the first and third quadrant.

We would like to optimize linear functions over S, that is, to understand conv S.

Linearize: replace each nonlinear monomial by a new variable.

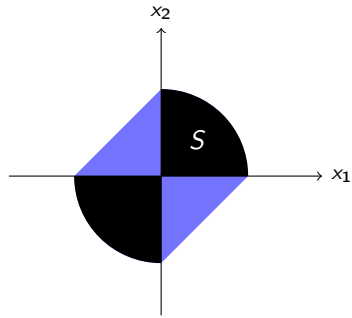
The solution set is now a polyhedron in $\mathbb{R}^5$. But its projection to the (x_1, x_2)-plane is the whole plane.



System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lccccc}
A & -x_1^2 & & -x_2^2 & +1 & \geq 0 \\
B & & x_1x_2 & & & \geq 0
\end{array}
$$



System of polynomial inequalities

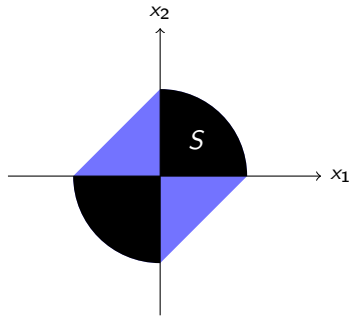
Less naive linearization

$$
\begin{array}{lrrrl}
A & -x_1^2 & & -x_2^2 & +1 \geq 0 \\
B & & x_1x_2 & & \geq 0
\end{array}
$$

redundant:

$$
D_1^2 \qquad x_1^2 - 2x_1x_2 + x_2^2 - 2x_1 + 2x_2 + 1 \geq 0
$$

Squares of $D_{1,2} := x_1 - x_2 \mp 1$, $D_{3,4} := 1 \pm x_1$, $D_{5,6} := 1 \pm x_2$.

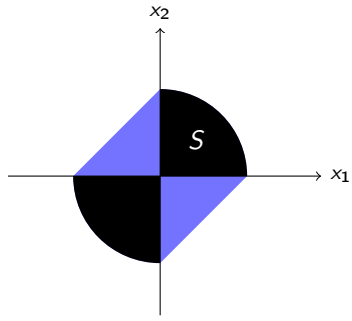


System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrl}
A & -x_1^2 & & & -x_2^2 & & & +1 & \geq 0 \\
B & & & x_1x_2 & & & & & \geq 0 \\
\text{redundant:} & & & & & & & & \\
D_1^2 & x_1^2 & -2x_1x_2 & & +x_2^2 & -2x_1 & +2x_2 & +1 & \geq 0 \\
D_2^2 & x_1^2 & -2x_1x_2 & & +x_2^2 & +2x_1 & -2x_2 & +1 & \geq 0
\end{array}
$$

Squares of $D_{1,2} := x_1 - x_2 \mp 1$, $D_{3,4} := 1 \pm x_1$, $D_{5,6} := 1 \pm x_2$.



System of polynomial inequalities

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$$
\begin{array}{lrrrrrr}
A & -x_1^2 & & -x_2^2 & & +1 & \geq 0 \\
B & & x_1x_2 & & & & \geq 0 \\
\text{redundant:} & & & & & & \\
D_1^2 & x_1^2 & -2x_1x_2 & +x_2^2 & -2x_1 & +2x_2 & +1 \geq 0 \\
D_2^2 & x_1^2 & -2x_1x_2 & +x_2^2 & +2x_1 & -2x_2 & +1 \geq 0 \\
D_{3,4}^2 & x_1^2 & & & \pm 2x_1 & & +1 \geq 0 \\
D_{5,6}^2 & & & x_2^2 & \pm 2x_2 & & +1 \geq 0
\end{array}
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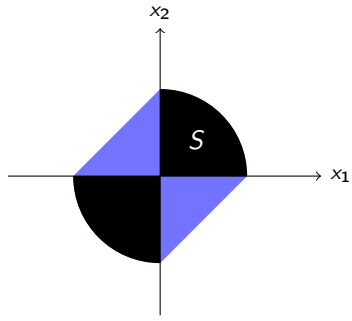
Squares of $D_{1,2} := x_1 - x_2 \mp 1$, $D_{3,4} := 1 \pm x_1$, $D_{5,6} := 1 \pm x_2$.



System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrr}
A & -x_1^2 & & -x_2^2 & & +1 & \geq 0 \\
B & & x_1x_2 & & & & \geq 0 \\
\text{irredundant:} & & & & & & \\
D_1^2 & x_1^2 & -2x_1x_2 & +x_2^2 & -2x_1 & +2x_2 & +1 \geq 0 \\
D_2^2 & x_1^2 & -2x_1x_2 & +x_2^2 & +2x_1 & -2x_2 & +1 \geq 0 \\
D_{3,4}^2 & x_1^2 & & & \pm 2x_1 & & +1 \geq 0 \\
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\end{array}
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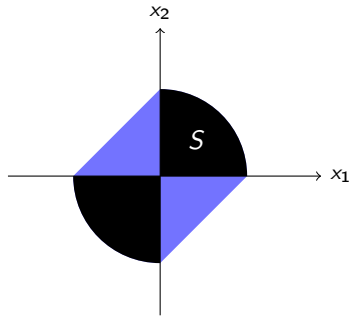

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lllllllll}
A & -\,y_1 & & -\,x_2^2 & & & & +\,1 & \geq 0 \\
B & & x_1x_2 & & & & & & \geq 0 \\
\text{irredundant:} & & & & & & & & \\
D_1^2 & y_1 & -\,2x_1x_2 & +\,x_2^2 & -\,2x_1 & +\,2x_2 & & +\,1 & \geq 0 \\
D_2^2 & y_1 & -\,2x_1x_2 & +\,x_2^2 & +\,2x_1 & -\,2x_2 & & +\,1 & \geq 0 \\
D_{3,4}^2 & y_1 & & & \pm & 2x_1 & & +\,1 & \geq 0 \\
D_{5,6}^2 & & & x_2^2 & \pm & 2x_2 & & +\,1 & \geq 0
\end{array}
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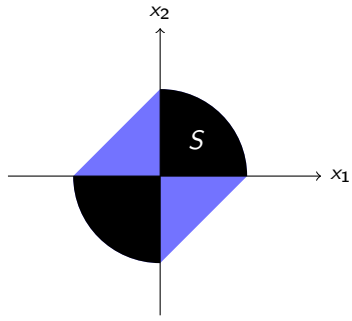

System of polynomial inequalities

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\begin{array}{lrrrrrrrl}
A & -\,y_1 & & -\,y_2 & & & & +\,1 & \geq 0 \\
B & & x_1x_2 & & & & & & \geq 0 \\
\text{irredundant:} & & & & & & & & \\
D_1^2 & y_1 & -\,2x_1x_2 & +\,y_2 & -\,2x_1 & +\,2x_2 & & +\,1 & \geq 0 \\
D_2^2 & y_1 & -\,2x_1x_2 & +\,y_2 & +\,2x_1 & -\,2x_2 & & +\,1 & \geq 0 \\
D_{3,4}^2 & y_1 & & & \pm & 2x_1 & & +\,1 & \geq 0 \\
D_{5,6}^2 & & & y_2 & \pm & 2x_2 & & +\,1 & \geq 0
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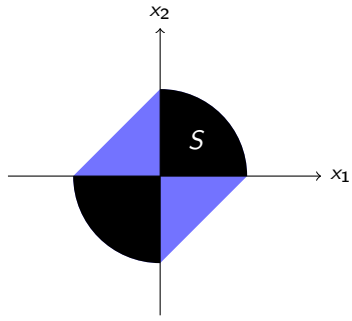
Squares of $D_{1,2} := x_1 - x_2 \mp 1$, $D_{3,4} := 1 \pm x_1$, $D_{5,6} := 1 \pm x_2$.



System of polynomial inequalities

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$$
\begin{array}{lrrrrrl}
A & -y_1 & & -y_2 & & +1 & \geq 0 \\
B & & y_3 & & & & \geq 0 \\
\text{irredundant:} & & & & & & \\
D_1^2 & y_1 & -2y_3 & +y_2 & -2x_1 & +2x_2 & +1 \geq 0 \\
D_2^2 & y_1 & -2y_3 & +y_2 & +2x_1 & -2x_2 & +1 \geq 0 \\
D_{3,4}^2 & y_1 & & & \pm 2x_1 & & +1 \geq 0 \\
D_{5,6}^2 & & & y_2 & \pm 2x_2 & & +1 \geq 0
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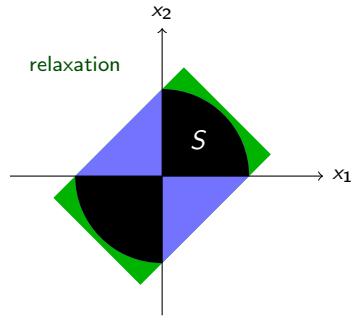

Now the projection is a polygon. Two of its edges are already right.

System of polynomial inequalities

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\begin{array}{lrrrrrl}
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B & & y_3 & & & & \geq 0 \\
\text{irredundant:} & & & & & & \\
D_1^2 & y_1 & -2y_3 & +y_2 & -2x_1 & +2x_2 & +1 \geq 0 \\
D_2^2 & y_1 & -2y_3 & +y_2 & +2x_1 & -2x_2 & +1 \geq 0 \\
D_{3,4}^2 & y_1 & & & \pm 2x_1 & & +1 \geq 0 \\
D_{5,6}^2 & & & y_2 & \pm 2x_2 & & +1 \geq 0
\end{array}
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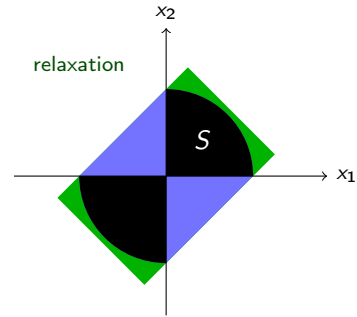
Squares of $D_{1,2} := x_1 - x_2 \mp 1$, $D_{3,4} := 1 \pm x_1$, $D_{5,6} := 1 \pm x_2$.

For example $A + 2B + D_1^2$: $2 - 2x_1 + 2x_2 \geq 0$.



Now the projection is a polygon. Two of its edges are already right.

System of polynomial inequalities

Less naive linearization

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B & & y_3 & & & \geq 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrr}
D_1^2 & y_1 & -2y_3 & +y_2 & -2x_1+2x_2 & +1 \geq 0 \\
D_2^2 & y_1 & -2y_3 & +y_2 & +2x_1-2x_2 & +1 \geq 0 \\
D_{3,4}^2 & y_1 & & \pm & 2x_1 & +1 \geq 0 \\
D_{5,6}^2 & & & y_2 \pm & 2x_2 & +1 \geq 0
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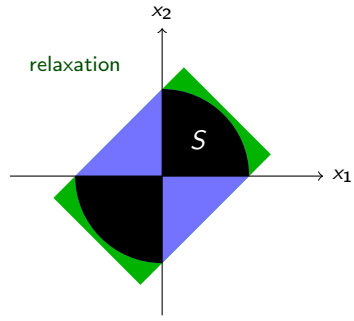
Squares of $D_{1,2} := x_1 - x_2 \mp 1$, $D_{3,4} := 1 \pm x_1$, $D_{5,6} := 1 \pm x_2$.

For example $A + 2B + D_1^2$: $2 - 2x_1 + 2x_2 \geq 0$.

Products like $A \cdot B$ are allowed too. They need further new variables.



Now the projection is a polygon. Two of its edges are already right.

No finite choice gives the round part.

System of polynomial inequalities

Systematic linearization

A whole family of redundant inequalities, parametrized by a, b, c:

$$(a + bx_1 + cx_2)^2 \geq 0$$

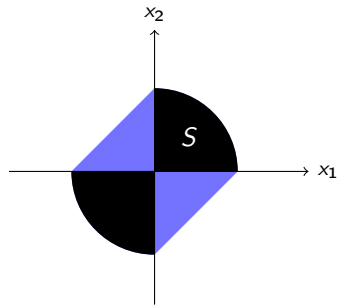
System of polynomial inequalities

Systematic linearization

A whole family of redundant inequalities, parametrized by a, b, c:

$$(a + bx_1 + cx_2)^2 \geq 0 \quad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$



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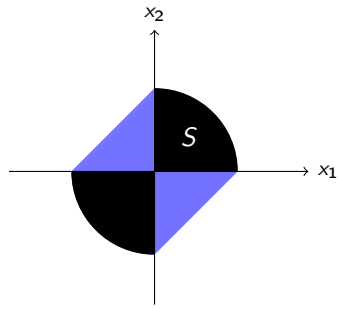
$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1 x_2 \\ x_2 & x_1 x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$



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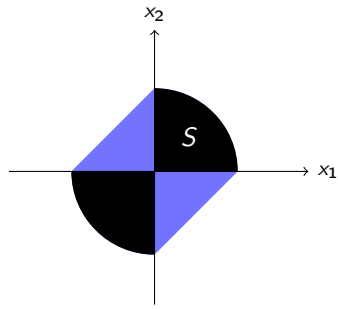

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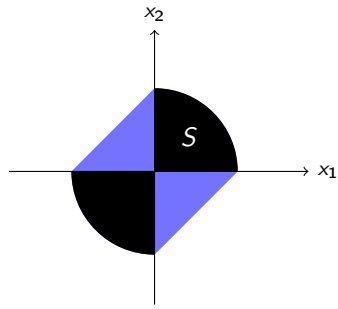
$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & y_1 & x_1 x_2 \\ x_2 & x_1 x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$



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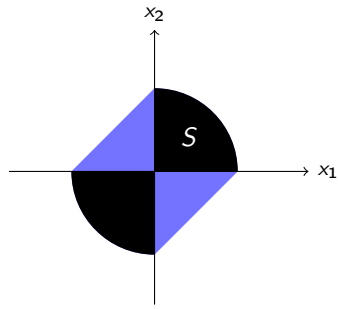


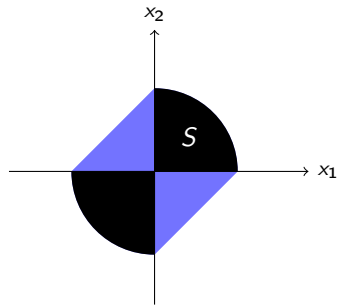
System of polynomial inequalities

Systematic linearization

A whole family of redundant inequalities, parametrized by a, b, c:

$$\begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & y_1 & y_3 \\ x_2 & y_3 & y_2 \end{pmatrix} \succeq 0$$

This is a single linear matrix inequality. Together with $1 - y_1 - y_2 \geq 0$ and $y_3 \geq 0$ it defines a spectrahedron in $\mathbb{R}^5$.



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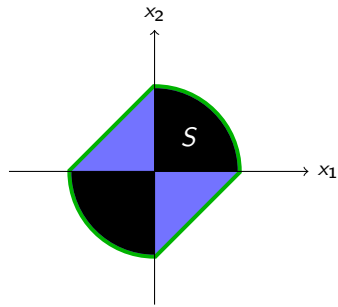
Its projection is exactly conv S.



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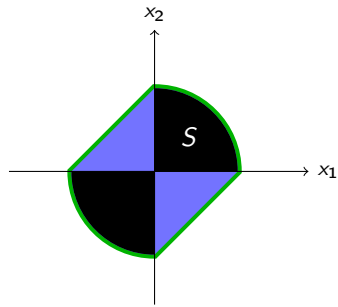

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Its projection is exactly conv S.

In general, for $\underline{g} = (g_1, \ldots, g_m)$ with $g_i \in \mathbb{R}[\underline{X}] := \mathbb{R}[X_1, \ldots, X_n]$, $g_0 := 1$ and a relaxation degree d: add all $p^2 g_i \geq 0$ of degree $\leq d$, linearize, and project.

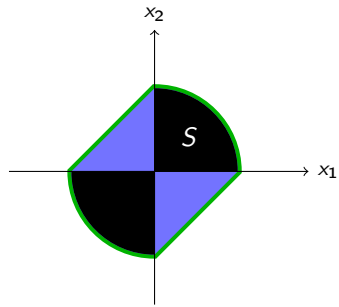
$$M_d(\underline{g}) := \left\{ \sum_{i=0}^{m} \sum_j p_{ij}^2 g_i \;\middle|\; \deg(p_{ij}^2 g_i) \leq d \right\} \qquad S_d(\underline{g}) := \text{ the projection (degree } d \text{ Lasserre relaxation)}$$

System of polynomial inequalities

Systematic linearization

A whole family of redundant inequalities, parametrized by a, b, c:

$$\begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & y_1 & y_3 \\ x_2 & y_3 & y_2 \end{pmatrix} \succeq 0$$



This is a single linear matrix inequality. Together with $1 - y_1 - y_2 \geq 0$ and $y_3 \geq 0$ it defines a spectrahedron in $\mathbb{R}^5$.

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$$S(\underline{g}) \subseteq \operatorname{conv} S(\underline{g}) \subseteq \ \ldots \ \subseteq S_{d+1}(\underline{g}) \subseteq S_d(\underline{g}) \subseteq \ \ldots$$

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Theorem (Putinar). Let $M(\underline{g}) := \bigcup_d M_d(\underline{g})$ be Archimedean. If $f > 0$ on $S(\underline{g})$, then $f \in M(\underline{g})$.

Exactness is a uniform degree bound for linear polynomials

Fact. Let $S(\underline{g})$ be compact and $d \in \mathbb{N}_0$. If every linear $\ell \in \mathbb{R}[\underline{X}]_1$ with $\ell \geq 0$ on $S(\underline{g})$ lies in $M_d(\underline{g})$, then

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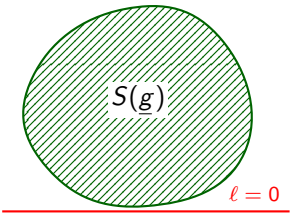

Warning. In Putinar's theorem, "$>$" cannot be replaced by "$\geq$". It says nothing about an ℓ that touches.

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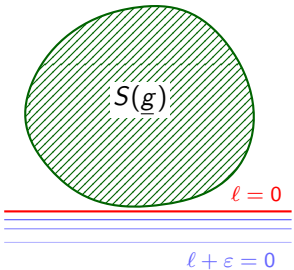

Warning. Approximating does not help: $\ell + \varepsilon \in M_{d(\varepsilon)}(\underline{g})$ for each real $\varepsilon > 0$, but $d(\varepsilon)$ may explode when $\varepsilon \to 0$.

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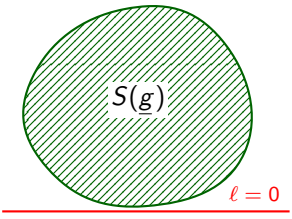
So the whole problem is: certificates with zeros for all supporting hyperplanes of $S(\underline{g})$, of one common degree d.



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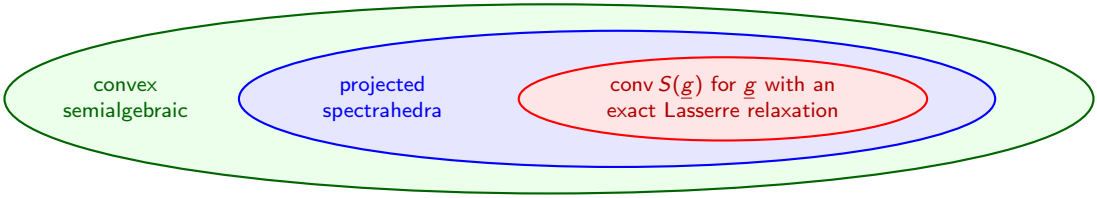

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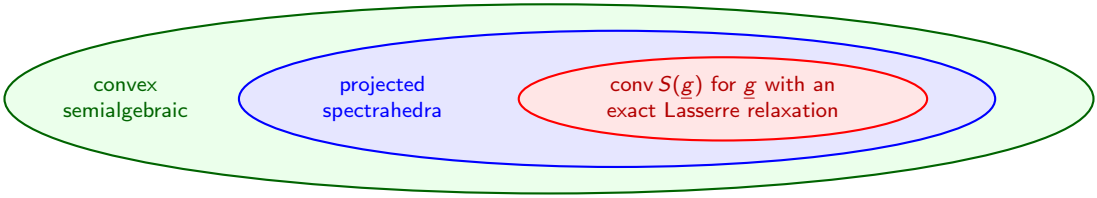

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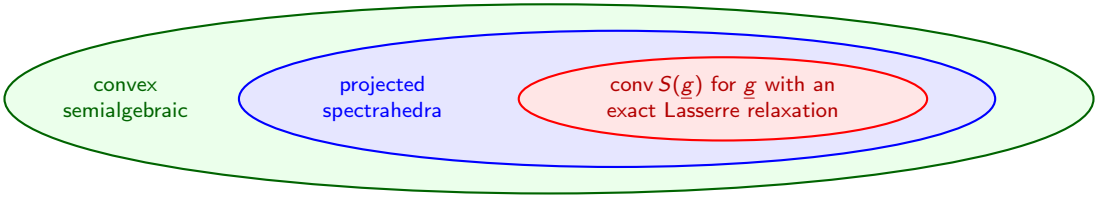

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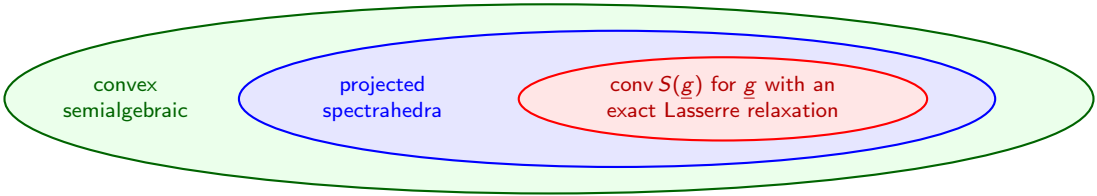

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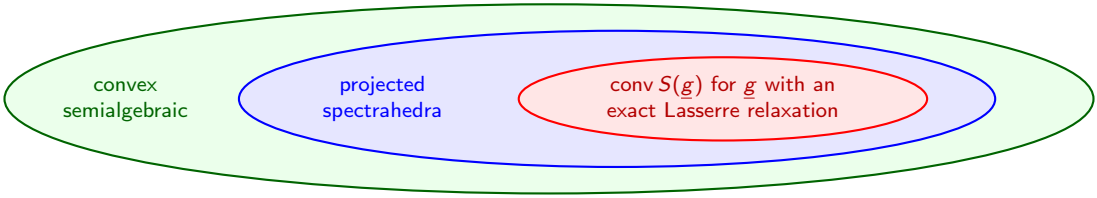

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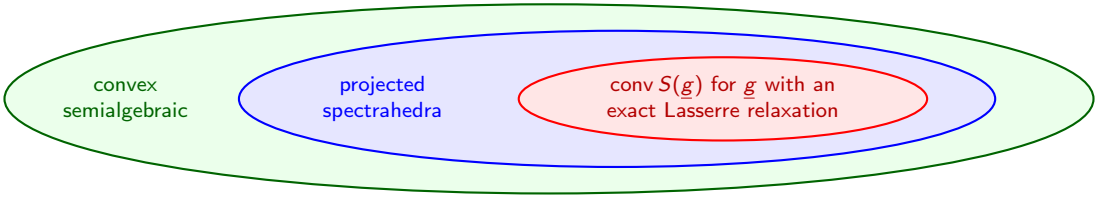

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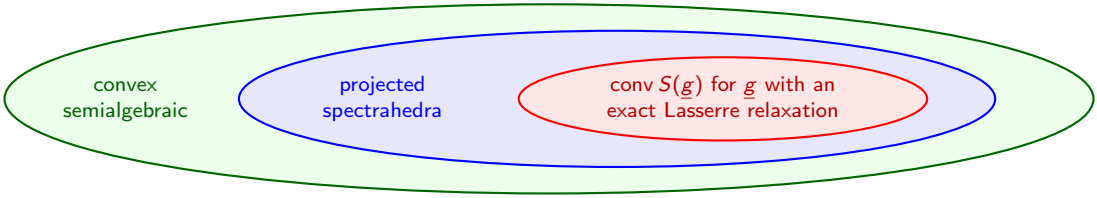

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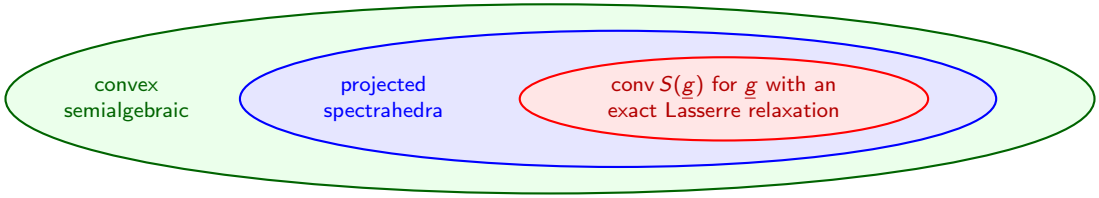

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Exactness asks for more: the Lasserre relaxation of the original inequalities has to do the job. This depends on the description $\underline{g}$ and not only on the set.

Two theorems

$h \in \mathbb{R}[\underline{X}]$ is strictly quasiconcave at x if $v^T(\operatorname{Hess} h)(x)v < 0$ for all $v \neq 0$ with $(\nabla h)(x)^T v = 0$. Then $\{h \geq 0\}$ looks near a boundary point x like a piece of a disk.

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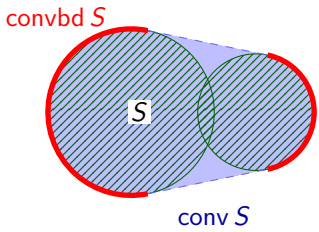
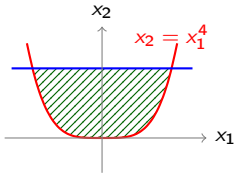
A is proved over $\mathbb{R}$ and allows linear constraints. B needs a real closed field and forbids them. Neither proof gives the other theorem.



Only the red part of the boundary matters: there S has to "bulge outwards".

Three examples in the plane



$X_2 - X_1^4 \geq 0,\ 1 - X_2 \geq 0$

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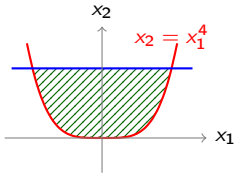

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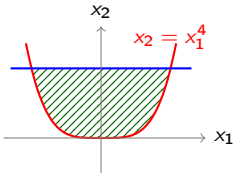

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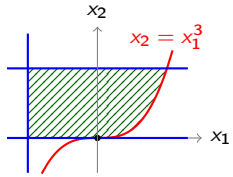

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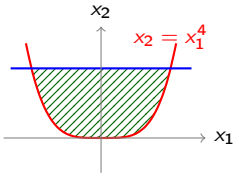

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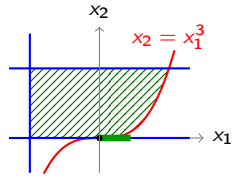

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Not exact: every S_d sticks out to the right of the origin (criterion of Gouveia & Netzer).

Yet it is a projected spectrahedron (Scheiderer, $n = 2$).

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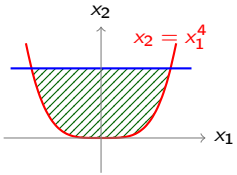

$X_2 - X_1^4 \geq 0,\ 1 - X_2 \geq 0$

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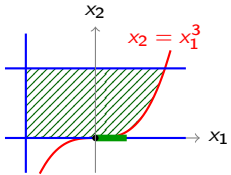

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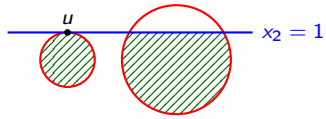

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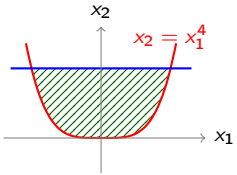

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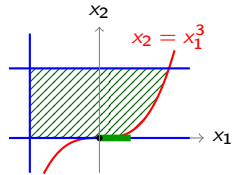

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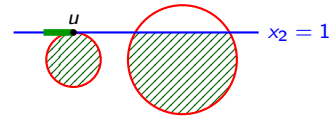

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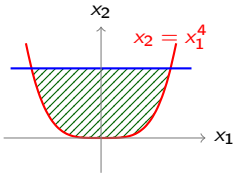

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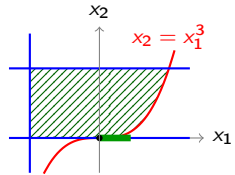

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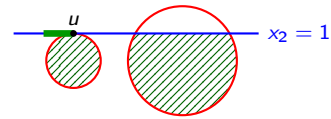

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In the non-convex world a single linear inequality can be fatal. The hypotheses of Theorem B are no artifact.

Convex versus non-convex over $\mathbb{R}$

Goal: $\exists d \;\; \forall \ell \in \mathbb{R}[\underline{X}]_1$ touching $S(\underline{g}) : \;\; \ell \in M_d(\underline{g})$

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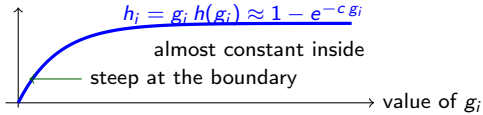

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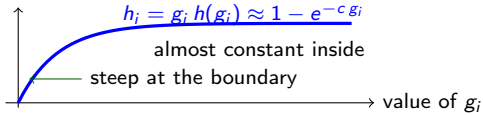

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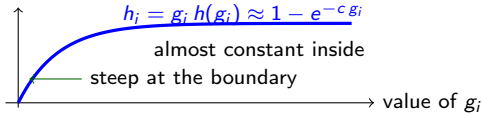

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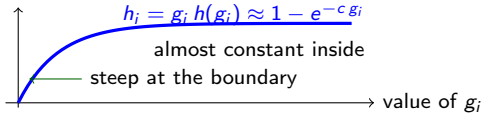

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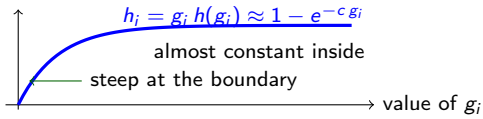

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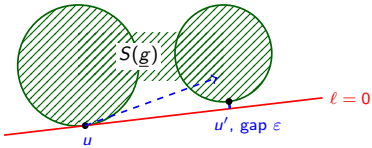

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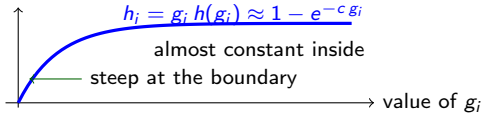

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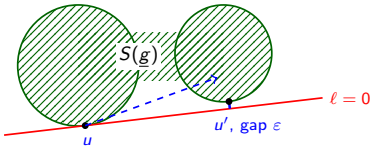

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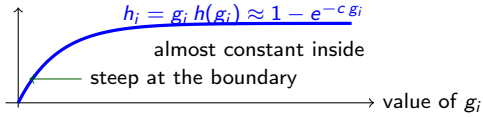

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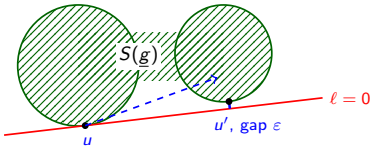

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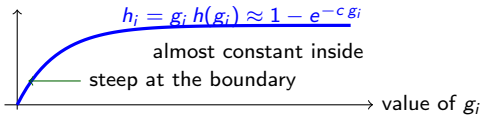

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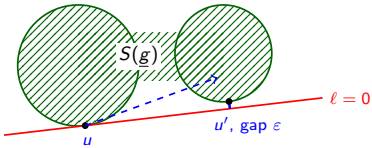

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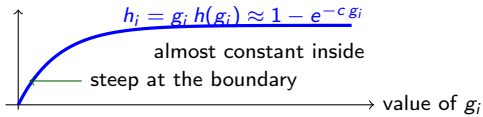

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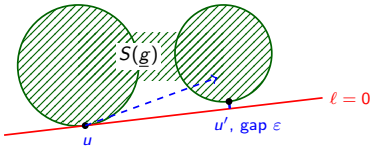
Each single ℓ still has a certificate. The trouble is uniformity in ℓ.



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Real closed fields and two tools from logic

A field R is real closed if $a \leq b \ :\Longleftrightarrow \ \exists c \in R : \ a + c^2 = b$ defines a linear order on R for which the intermediate value theorem for polynomials holds. From now on, R is a real closed extension field of $\mathbb{R}$.

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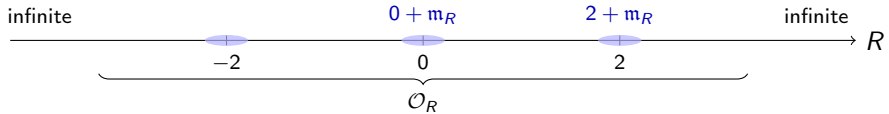

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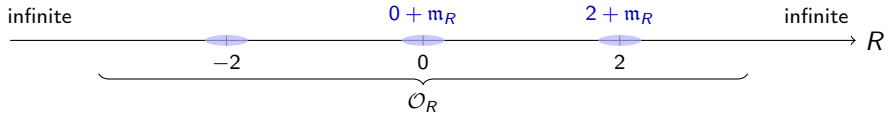

Tarski's transfer principle. A semialgebraic set $S \subseteq \mathbb{R}^n$ extends canonically to $S_R \subseteq R^n$. A first-order statement about semialgebraic sets holds over $\mathbb{R}$ if and only if it holds over R.

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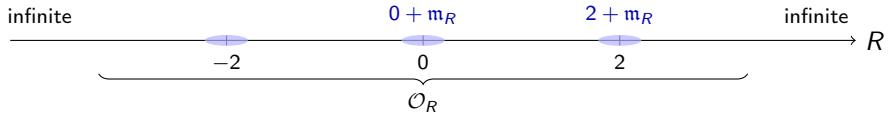

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Finiteness theorem. Let $S_1 \subseteq S_2 \subseteq \ldots$ be semialgebraic subsets of $\mathbb{R}^N$. If $\bigcup_{d\in\mathbb{N}} (S_d)_R = R^N$ for *all* R, then $S_d = \mathbb{R}^N$ for some d. This is the compactness theorem of first-order logic in disguise.

Effect 1: the degree disappears

Goal over $\mathbb{R}$: $\exists d \;\; \forall \ell \in \mathbb{R}[\underline{X}]_1 : \; (\ell \geq 0 \text{ on } S(\underline{g}) \implies \ell \in M_d(\underline{g}))$

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For $d \in \mathbb{N}$, let $S_d \subseteq \mathbb{R}^{n+1}$ be the set of all coefficient vectors $(a_0, \ldots, a_n)$ such that

$$\ell := a_0 + a_1 X_1 + \cdots + a_n X_n \geq 0 \text{ on } S(\underline{g}) \implies \ell \text{ is a sum of } d \text{ terms } p^2 g_i \text{ of degree} \leq d.$$

Each S_d is semialgebraic (real quantifier elimination) and $S_1 \subseteq S_2 \subseteq \ldots$

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Each S_d is semialgebraic (real quantifier elimination) and $S_1 \subseteq S_2 \subseteq \ldots$

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over $\mathbb{R}$	over R
supporting hyperplanes ℓ_k converging to one with more contact points	one supporting $\ell \in \mathcal{O}_R[\underline{X}]_1$; $\mathrm{st}(\ell)$ has more contact points than ℓ
gaps $\varepsilon_k \to 0$ at the lost contact points, degrees d_k	gaps in $\mathfrak{m}_R$, no degree

A wrong theorem

What one might write down when generalizing Putinar's theorem to real closed fields

Wrong Theorem. Let $M(\underline{g})$ be Archimedean and let R be a real closed extension field of $\mathbb{R}$. If $f \in R[\underline{X}]$ satisfies $f > 0$ on $S(\underline{g})_R$, then f lies in the quadratic module generated by $\underline{g}$ in $R[\underline{X}]$.

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Positivity has to be measured against the real numbers, not inside R. Only the Archimedean hull $\mathcal{O}_R$ of $\mathbb{R}$ in R comes to our rescue.

Effect 2: zeros that are infinitesimally displaced

Test states (S. & Vargas 2024, SIAM J. Appl. Algebra Geom.). Putinar tests f with point evaluations. At a zero a of f one tests instead with linear maps φ on an ideal $I \ni f$ such that $\varphi(pq) = p(a)\varphi(q)$ for $p \in \mathbb{R}[X]$, $q \in I$: typically derivatives at a.

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Key Lemma (Kriel & S. 2019). Let M be an Archimedean quadratic module of $\mathcal{O}[\underline{X}]$ and $S := \{x \in \mathbb{R}^n \mid \forall p \in M : \mathrm{st}(p(x)) \geq 0\}$. Let $x_1, \ldots, x_k \in \mathcal{O}^n$ have pairwise distinct standard parts. Let $f \in I_{x_1}^2 \cap \cdots \cap I_{x_k}^2$ with

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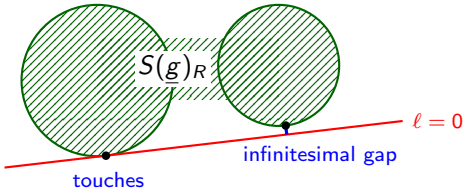
- The tests are still real-valued: linear maps φ on $\bigcap_i I_{x_i}^2$ with $\varphi(pq) = \mathrm{st}(p(x_i))\,\varphi(q)$. They are second derivatives at the displaced points, followed by the standard part.

Effect 3: several almost-contact points at once



A supporting hyperplane over R touches $S(\underline{g})_R$ at one place. At other places it may come infinitesimally close without touching.

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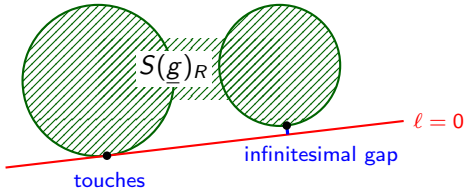

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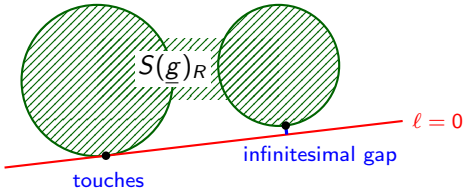

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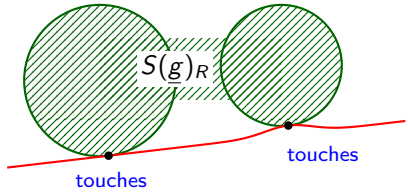

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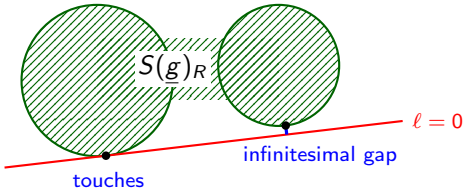

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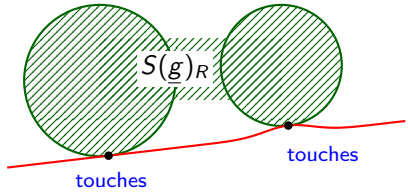

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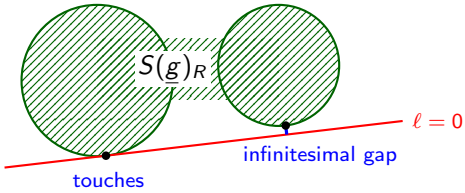

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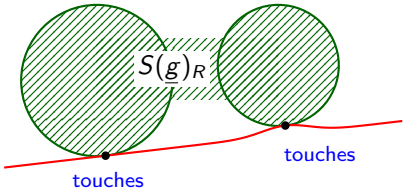

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With one almost-contact point this would be easy. For non-convex sets there are several, but only finitely many.

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Real closed fields help when you need a bound that is uniform over a family that degenerates.
Open: a common generalization of Theorems A and B.