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A constructive approach to Putinar's and Schmüdgen's Positivstellensätze with applications to degree bounds and matrix polynomials

In the year 1900, Hilbert presented a list of 23 very influential mathematical problems [33]. In the 17th of these problems he mainly asked whether each (globally) nonnegative (real) polynomial (in several variables) could be written as a sum of squares of rational functions (already knowing that sums of squares of polynomials are not enough [34, page 347], i.e. denominators are needed). Artin solved this problem to the affirmative [31, Satz 6][8, 1.4.1][18, Thm. 2.1.12]. To do this, he had to develop together with Schreier the theory of ordered fields [32]. In particular, he had to introduce the notion of a real closed field [32, p. 87]. Real closed fields relate to ordered fields very much like algebraically closed fields relate to fields. The real numbers form the prototype of a real closed field just like the complex numbers build the prototype of an algebraically closed field. Real closed fields seem to be an indispensable tool for answering Hilbert's 17th problem. Moreover, the answer to Hilbert's 17th problem remains positive over an arbitrary real closed field (i.e. when one allows for coefficients from a real closed field instead of the real numbers). By general arguments from model theory, this implies that the degree of the numerator and denominator of the rational functions in a sums of squares representation of a given nonnegative polynomial can be bounded in terms of the number of variables and the degree of this polynomial. To get a concrete bound however is extremely tedious (and the known bounds are horribly bad) since Artin's proof is highly non-constructive [23, 24].

At about the same time when Artin solved Hilbert's 17th problem, Pólya proved another theorem on positive polynomials of a totally different flavor. He characterized (real) homogeneous polynomials (strictly) positive on an open orthant. Namely, he showed that these can be written as a quotient of an homogeneous polynomial with only (strictly) positive coefficients and a power of the sum of the variables [30]. For the case of two variables this is easily seen to be equivalent to the fact that a univariate polynomial positive on a given interval has only positive coefficients when expressed in the Bernstein basis of the vector space of polynomials of sufficiently high degree (associated to the interval). In sharp contrast to Artin's theorem, in Pólya's theorem it is self-evident how to compute the guaranteed representation: One simply multiplies the polynomial repeatedly with the sum of the variables until all coefficients get positive. Powers and Reznick proved an upper bound for the number of repetitions needed [17]. This bound unfortunately depends on a measure of how close the polynomial gets to zero (loosely speaking the size of the coefficients divided by the minimum on the standard simplex). Pólya's theorem does not hold over any real closed field.

Just a few years after the discovery of Artin's and Pólya's theorems, Tarski invented the method of real quantifier elimination. This was published only about

20 years later [29]. Then another thirteen years passed before this tool which is omnipresent in modern real algebra showed its impact on the further developments around Hilbert's 17th problem.

Namely, the next major step in the theory of positive polynomials was Krivine's seminal paper [28] where he introduces amongst others preordered rings and their maximal real spectrum. Thus he introduced basic notions and indispensable tools of modern real algebraic geometry in his very first scientific article. This brilliant work came too early for people to understand what is going on and has been neglected for a long time [18, Section 4.7]. Combining this newly developed theory with Tarski's real quantifier elimination, Krivine proves in the same article both the Positivstellensatz [28, Thm. 7][18, 4.2.10][21, 4.4.2] and the archimedean Positivstellensatz [18, Lemma 5.2.7].

The Positivstellensatz is a refinement and generalization of Hilbert's 17th problem which might at first glance look like a technical improvement but actually it is a very crucial and important enhancement of Artin's theorem. The archimedean Positivstellensatz at first sight looks like a variant of the Positivstellensatz but actually it is more of a Pólya-like nature: Under the additional assumption that the given preorder is archimedean, it provides a concrete denominator (namely a natural number which in many rings can be assumed to be 1). However, Krivine uses the Positivstellensatz to prove the archimedean Positivstellensatz (combine [28, Thm. 11] with [28, Thm. 7]). Note however, that in the archimedean setting one can easily avoid the use of Tarski's theorem to prove the Positivstellensatz. Krivine's proof of the archimedean Positivstellensatz is completely constructive up to the starting point of the proof where he applies the Positivstellensatz to the element one wants to represent. Much later the author of this note gives a different and completely constructive proof of the archimedean Positivstellensatz by reducing it to Pólya's Theorem [16] (see also [1] for a recent exposition). We will come back to this later.

The content Krivine's work was disremembered for about 35 years (though the work has occasionally been cited even in [21, page 95]) until Prestel took notice of this. Even now it continues to be ignored by many authors. Therefore the Positivstellensatz is often attributed to Stengle who rediscovered it ten years later [27]. Independantly, Prestel rediscovered at about the same time the Positivstellensatz [26, Thm. 5.10] and gave the modern standard proof.

Unexpectedly, the next major breakthrough in the theory of positive polynomials came from functional analysis. In 1991, Schmüdgen used the Positivstellensatz to prove that multiplication operators arising in a GNS construction are bounded and used the spectral theorem and separating techniques for convex sets to prove what is now the celebrated Schmüdgen's Positivstellensatz [25, Cor. 3][18, Thm. 5.2.9][8, Cor. 6.1.2]. It is a denominator-free version of the Positivstellensatz over compact semialgebraic sets. It took more than seven years until people from real algebraic geometry could find an algebraic proof for Schmüdgen's Positivstellensatz. Namely, Wörmann found in his thesis (see [19]) an amazingly short but ingenious algebraic argument that allows to deduce Schmüdgen's Positivstellensatz from the Positivstellensatz and the archimedean Positivstellensatz. Using the Positivstellensatz Wörmann could show that the preorder involved in Schmüdgen's Positivstellensatz is archimedean and fulfills therefore the hypotheses of the archimedean Positivstellensatz. In hindsight, Schmüdgen's theorem is thus a characterization of finitely generated archimedean preorderings in the real polynomial ring [18, Thm. 5.1.17][8, Thm. 6.1.1] rather than a theorem about positive polynomials. But the original proof worked very differently. In the original proof there is a gap reported by Marshall in [8, pages x,88,89 and 98]. This gap has been found by Prestel and shortly after it has been bridged by Schmüdgen in an unpublished erratum which was apparently not known to Marshall.

Just two years later, in 1993, Putinar proved also with functional-analytic methods a sharpening of the archimedean Positivstellensatz which is now known as Putinar's Positivstellensatz [22][18, Thm. 5.3.8]. He uses quadratic modules instead of preorderings. The sums of squares representation is therefore weighted only by the defining polynomials of the semialgebraic set instead of all their exponentially many products. It is a common misperception that Putinar's Positivstellensatz is a strengthening of Schmüdgen's Positivstellensatz. In fact, it is a strengthening of the archimedean Positivstellensatz although one could formulate it in a way that it would generalize at the same Schmüdgen's Positivstellensatz (by imposing condition (i) from [12, Thm. 1] instead of the archimedean condition as a hypothesis). But any such phrasing of Putinar's Positivstellensatz just borrows from Schmüdgen's characterization of archimedean preorderings which is much deeper than Putinar's theorem. In fact, the innovative aspect of Putinar's article was mostly something different and his Positivstellensatz was "just" a by-product. Nevertheless it took again more than seven years until people from real algebraic geometry could find an algebraic proof for Putinar's Positivstellensatz. It was Jacobi who found a very technical and long algebraic argument [18, Lemma 5.3.7]. Another seven years later Marshall found an ingenious argument that radically shortened Jacobi's proof [8, Thm. 5.4.4].

The author's constructive approach. In 2002, the author found a new proof of the archimedean Positivstellensatz which is completely constructive [13]. It uses Pólya's theorem instead of the Positivstellensatz. It is therefore also an algorithmic approach to the Positivstellensatz up to Schmüdgen's characterization of archimedean preorderings. The latter is still not constructive at all since it uses the Positivstellensatz and Tarski's real quantifier elimination (note that we said above that Tarski could be avoided in the Positivstellensatz in the presence of the archimedean condition, however this does not help since it is used at a point in the proof before the archimedean condition is established).

In 2005, the author found a similar approach to Putinar's Positivstellensatz. The constructions involved are much more "dirty" than for the archimedean Positivstellensatz in the sense that there is an additional step with a polynomial of potentially very large degree appearing even before Pólya's procedure is applied.

The main advantages of these constructive approaches are the following:

(1) Computation of sums of squares representations. One can actually try to compute the sums of squares representation in the archimedean Positivstellensatz or in Putinar's Positivstellensatz. Once a Positivstellensatz certificate for the archimedean property is known, this then applies also to Schmüdgen's Theorem. Such a certificate can be found in many cases, and one gets it for free by adding a redundant inequality defining a big ball to the description of the semialgebraic set (if a ball containing the set is known).

(2) Complexity analysis. By taking much more care in the constructions, one can take track of the degree complexity of the sums of squares representations in the archimedean Positivstellensatz and in Schmüdgen's Positivstellensatz [13]. One of the main ingredients is the upper bound on the exponent needed in Pólya's theorem proved by Powers and Reznick [17]. Therefore it is not surprising that again the bound depends on a measure of how close the polynomial gets to zero on the semialgebraic set (roughly speaking again the size of the coefficients divided by the minimum on the semialgebraic set).

The same is true for Putinar's Positivstellensatz [9]. However, the bound is considerably worse. It seems that the price one has to pay for avoiding the exponentially many products of the defining inequalities is an exponential in the degree bound (though it is not known if the bounds are sharp).

(3) Parameterized families of sums of squares representations. The constructions, if performed carefully enough, can often be done uniformly for parameterized families of polynomials to represent. For the final stage of the procedure, namely the repeated multiplication step in Pólya's theorem, to terminate, it is often advantageous if the parameters come from a compact space.

Applications. The applications of the author's procedure seem to be numerous and are by far not exhausted. We give here just a few examples.

A. Computing minima of polynomials on compact semialgebraic sets. One can try to get a sums of squares representation of a polynomial minus an unknown lower bound of the polynomial. After each multiplication step in Pólya's procedure, one solves a linear program in only two (!) variables with the objective of maximizing the unknown lower bound. The second variable in the linear program comes from a parameter introduced in the author's constructions. This is an example of (3) with a linearly parameterized family of polynomials, the parameter ranging over an interval of the real line (namely the set of strict lower bounds of the polynomial on the given semialgebraic set). This procedure was implemented by Datta [15].

B. Positive polynomials on cylinders with compact cross section. Powers had the idea to consider a polynomial on a cylinder with compact cross section as a parameterized family of polynomials on the same compact semialgebraic set (namely the cross section). In this way she found mild and reasonable geometric conditions that guarantee the existence of sums of squares representations of polynomials positive on such cyclinders [14]. This is again an example of (3) with a linearly parameterized family of polynomials, the parameter ranging over the whole real line.

C. Positive matrix polynomials. A symmetric matrix polynomials in several variables can be interpreted as a polynomial in the same variables with coefficients which are quadratic forms in new variables (one for each row or column). Since quadratic forms are given by their values on the unit sphere, one can therefore think of symmetric matrix polynomials as parameterized polynomials with parameters in the unit sphere which is a compact space. The ideas in (3) above therefore apply. This was carried out by Hol and Scherer in order to prove a version of Putinar's theorem for matrix polynomials [10] (see also [13]).

D. Semidefinite representations. In two seminal articles, Helton and Nie proved that many convex semialgebraic sets are semidefinitely representable [6, 5] (attention: the two articles appeared in the wrong order). To prove this they need sums of squares representation of bounded degree complexity for linear polynomials nonnegative on the given semialgebraic set. The main focus lies on the linear polynomials whose kernel is a supporting hyperplane of the convex set. Therefore neither Schmüdgen's nor Putinar's Positivstellensatz is applicable since the linear form is not strictly positive on the set. Although there are meanwhile a lot of theorems generalizing these theorems by allowing for a certain kind of zeros [11, 7, 2], there are no general complexity bounds available (perhaps one could try to generalize the author's constructive approach by using versions of Pólya's theorems that allow for zeros [3] but this seems a long way to go). Helton and Nie found a truly ingenious way to control the degree complexity by using Karush-Kuhn-Tucker conditions (i.e., "Lagrange-multipliers" for inequalities) and a sums of squares representation of the Hessian. The Hessian is a symmetric matrix polynomial which can very roughly speaking be assumed to be positive with some additional arguments given by Helton and Nie. This created the need for a matrix version of Schmüdgen's and Putinar's Positivstellensatz with control on the degree complexity. But with the observation made in the last point that matrix polynomials fall under the general idea (3) above, the arguments in [13, 9] go through almost literally as Helton and Nie observed.

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