Describing convex semialgebraic sets by linear matrix inequalities

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Introduction

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$\rightsquigarrow$ Chris Brown et al., Wednesday, Room B, 14:00 – 15:15

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If φ defines $S \subseteq \mathbb{R}^{n+m}$, then $\exists x_{m+1}, \dots, x_{n+m} \in \mathbb{R} : \varphi$ defines the image of S under the projection

$$\pi \colon \mathbb{R}^{n+m} \to \mathbb{R}^n,\ (x_1, \dots, x_n, \dots, x_{n+m}) \mapsto (x_1, \dots, x_n).$$

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Disregarding algorithmic issues, real quantifier elimination thus simply says that projections of semialgebraic sets are again semialgebraic.

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if $S = \{y \in \mathbb{R}^n \mid \varphi\}$.

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$$S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$$

for some $m \in \mathbb{N}$ and some polynomials $g_1, \ldots, g_m \in \mathbb{R}[\bar{X}]$.

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Here and throughout the talk $\bar{X} := (X_1, \ldots, X_n)$ is an n-tuple of variables and $\mathbb{R}[\bar{X}] := \mathbb{R}[X_1, \ldots, X_n]$ denotes the algebra of real polynomials in n variables.

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Can the number m of inequalities be bounded?

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A basic open semialgebraic set in $\mathbb{R}^n$ is the solution set of a finite system of strict polynomial inequalities.

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$$S = \{x \in \mathbb{R}^n \mid g_1(x) > 0, \dots, g_m(x) > 0\}$$

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Theorem (Bröcker & Scheiderer 1989).
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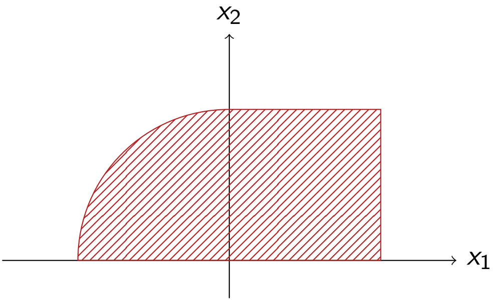
Example.

$S := (\{(x,y) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\} \cap ([-1,1] \times [0,1])) \cup [0,1]^2$ is closed and semialgebraic but not basic closed.



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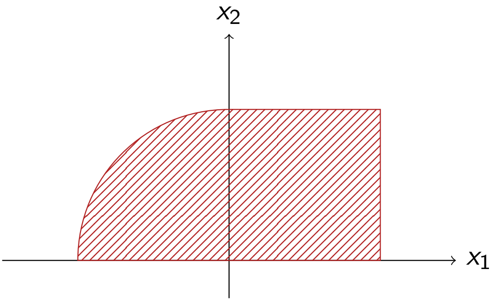

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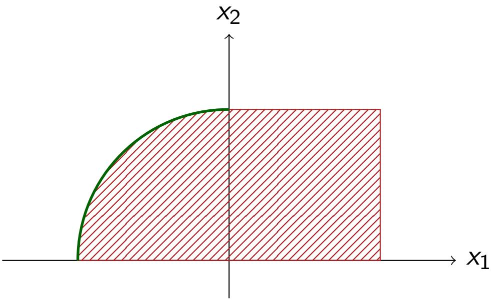

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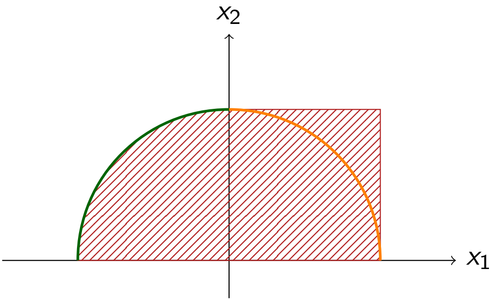
Looking at the green points, one of the g_i could be written as $g_i = h \cdot (1 - X_1^2 - X_2^2)^k$ for some odd $k \geq 1$ and $h \in \mathbb{R}[X_1, X_2]$ not divisible by $1 - X_1^2 - X_2^2$.
Looking at the orange points, h would be divisible by $1 - X_1^2 - X_2^2$.



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Convexity is a crucial feature in numeric computation (e.g., in interior point methods for convex optimization) but seems to be neglected in symbolic computation.

We think that other representations should be chosen for convex semialgebraic sets.

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$$\operatorname{conv} S = \left\{ \sum_{i=1}^{N} \lambda_i x_i \mid N \in \mathbb{N}, x_i \in S, \lambda_i \geq 0, \lambda_1 + \cdots + \lambda_N = 1 \right\}.$$

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if $S \subseteq \mathbb{R}^n$ is semialgebraic, then $\operatorname{conv} S$ is also semialgebraic.

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Note also that projections of convex sets are again convex.

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Proposition. Let $S \subseteq \mathbb{R}^n$ be convex. Then

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(b) If F_1, F_2 are faces of S and $F_1 \subsetneq F_2$, then $\dim F_1 < \dim F_2$.

(c) The intersection of any two faces of S is again a face of S.

A convex subset $F \neq \emptyset$ of a convex set S is called a face of S if any line segment $L \subseteq S$ whose relative interior intersects F is actually contained in F.

In particular: If $S \neq \emptyset$, then S is always a face of itself. Any other face of S is contained in the boundary of S. A singleton $F = \{x\}$ is a face of S if and only if x is an extreme point of S.

Proposition. Let $S \subseteq \mathbb{R}^n$ be convex. Then

(a) Any face of a face of S is a face of S.

(b) If F_1, F_2 are faces of S and $F_1 \subsetneq F_2$, then $\dim F_1 < \dim F_2$.

(c) The intersection of any two faces of S is again a face of S.

(d) S is the disjoint union of the relative interiors of its faces.

By a hyperplane, we understand here an affine linear subspace of codimension one in $\mathbb{R}^n$. Any hyperplane divides $\mathbb{R}^n$ into two closed or open half-spaces.

Describing convex semialgebraic sets by LMIs

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Let $S \subseteq \mathbb{R}^n$ be convex.

A supporting hyperplane of S is a hyperplane H such that $S \cap H \neq \emptyset$ and S is contained entirely in one of the two closed half-spaces determined by H.

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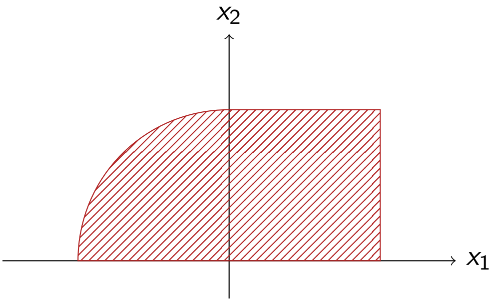
If H is a supporting hyperplane of S, then $S \cap H$ is a face of S. These faces as well as S itself are called exposed faces of S.

Describing convex semialgebraic sets by LMIs

Example. The faces of $S := (\overline{B_1(0)} \cap ([-1,1] \times [0,1])) \cup [0,1]^2$ are



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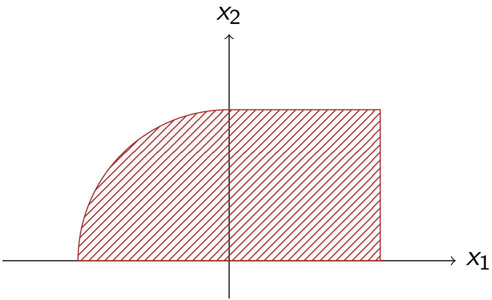

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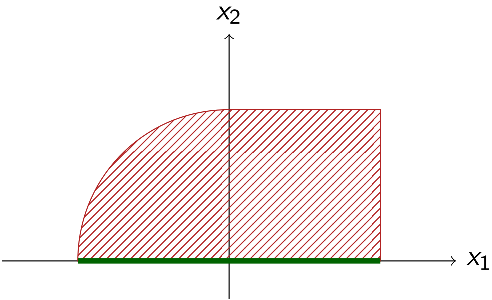

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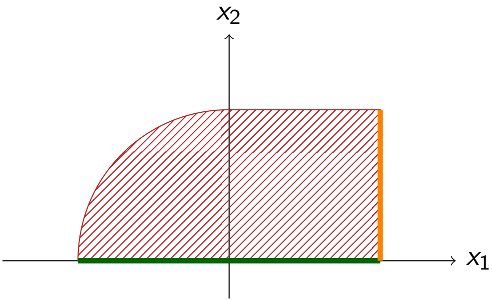

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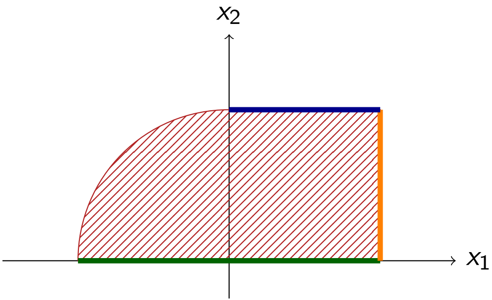

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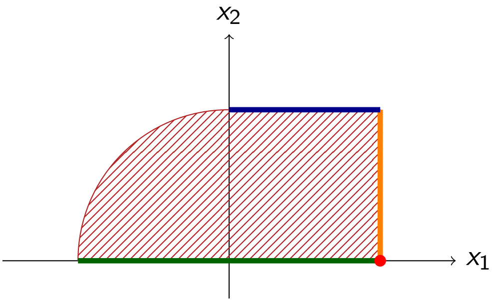

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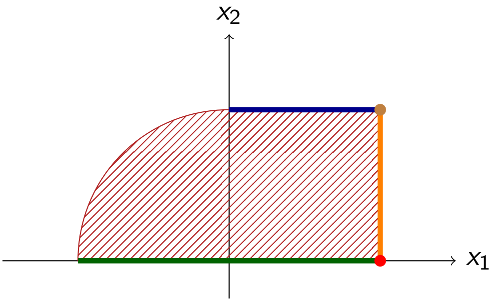

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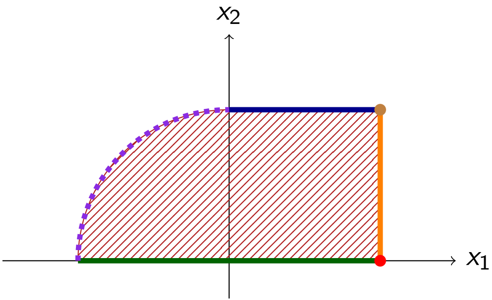

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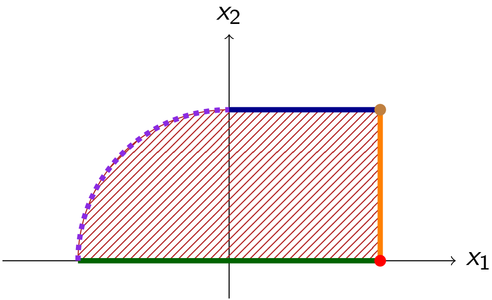

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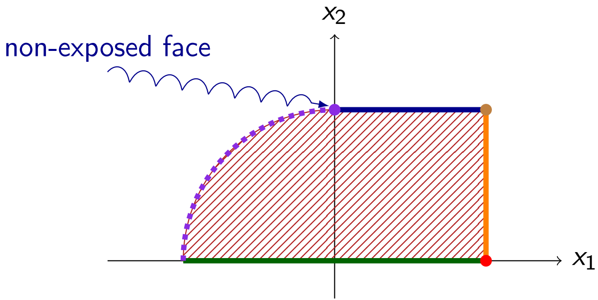

Describing convex semialgebraic sets by LMIs

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The degree of a matrix polynomial is the maximal degree of its entries.

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To define LMIs and for later use, we consider matrix polynomials (also called polynomial matrices), i.e., elements of $\mathbb{R}[\bar{X}]^{s \times t}$.

The degree of a matrix polynomial is the maximal degree of its entries. A linear matrix polynomial is a matrix polynomial of degree at most 1, i.e., of the form $A_0 + X_1 A_1 + \cdots + X_n A_n$ for matrices $A_i \in \mathbb{R}^{s \times t}$.

Describing convex semialgebraic sets by LMIs

Let $A \in S\mathbb{R}^{t \times t}$.

$$A \succeq 0 \quad \Longleftrightarrow \quad A \text{ positive semidefinite}$$

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\begin{aligned}
A \succ 0 \quad &\iff \quad A \text{ positive } \not{s}\not{e}\not{m}\not{i}\text{definite} \\
&\iff \quad \langle Av, v\rangle > 0 \text{ for all } v \in \mathbb{R}^t \setminus \{0\} \\
&\iff \quad \text{all eigenvalues of } A \text{ are } > 0 \\
&\iff \quad \text{all coefficients of } \det(A + TI_t) \in \mathbb{R}[T] \text{ are } > 0 \\
&\iff \quad \det((A_{ij})_{i,j\in J}) > 0 \text{ for all } J \subseteq \{1,\dots,t\} \\
&\iff \quad \det((A_{ij})_{i,j\in\{1,\dots,k\}}) > 0 \text{ for all } k \in \{1,\dots,t\}
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An inequality of the form

$$A(x) := A_0 + x_1 A_1 + \dots + x_n A_n \succeq 0 \qquad (x \in \mathbb{R}^n)$$

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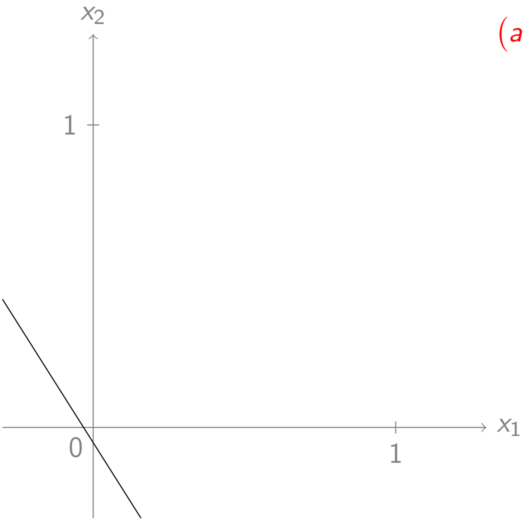
with $A_0,\ldots,A_n \in S\mathbb{R}^{t\times t}$ will be called linear matrix inequality.
This corresponds to the family of linear inequalities

$$\langle A(x)v, v\rangle \geq 0 \qquad (x \in \mathbb{R}^n)$$

parametrized by $v \in \mathbb{R}^t$.

Describing convex semialgebraic sets by LMIs



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a, b, c independant
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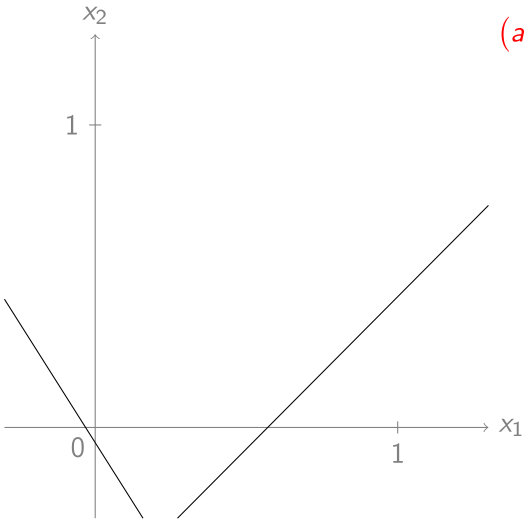

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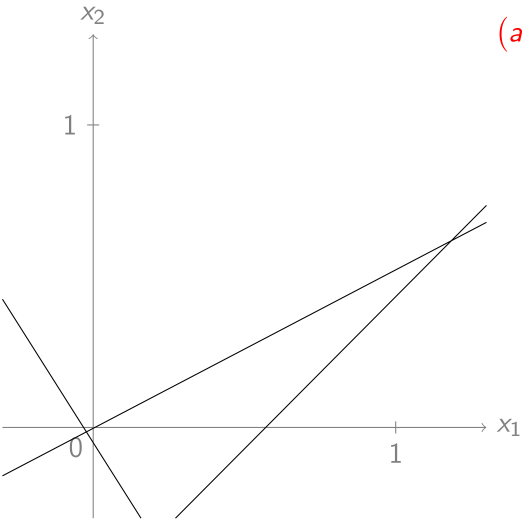

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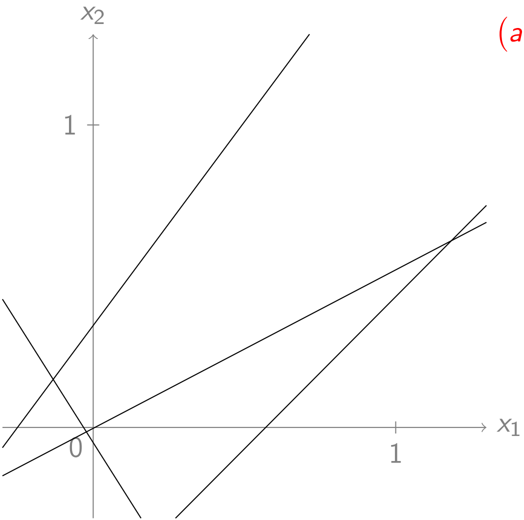

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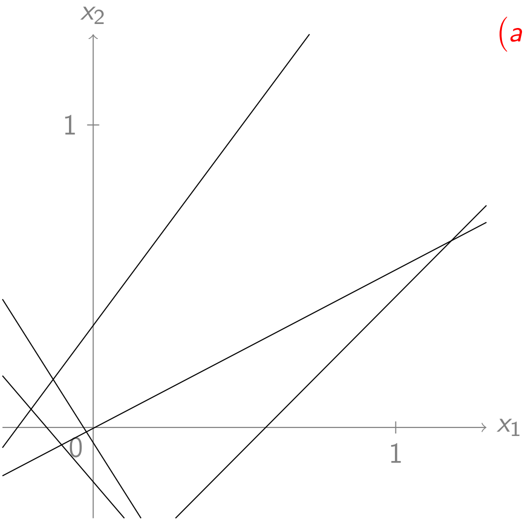

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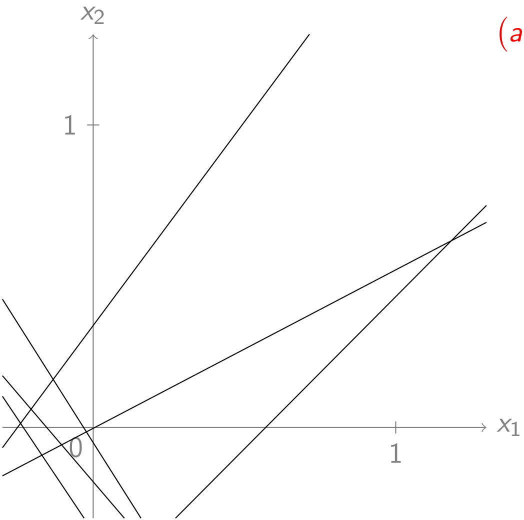

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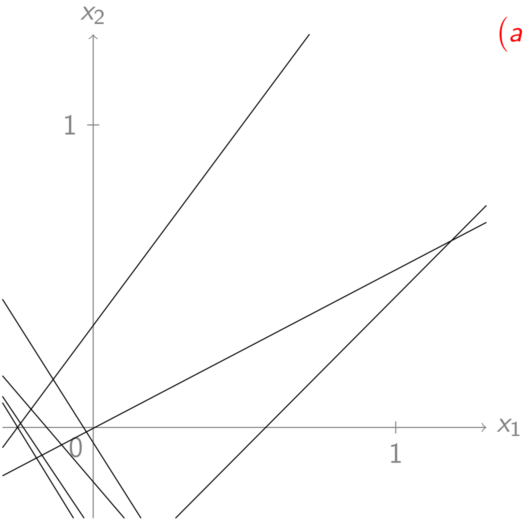

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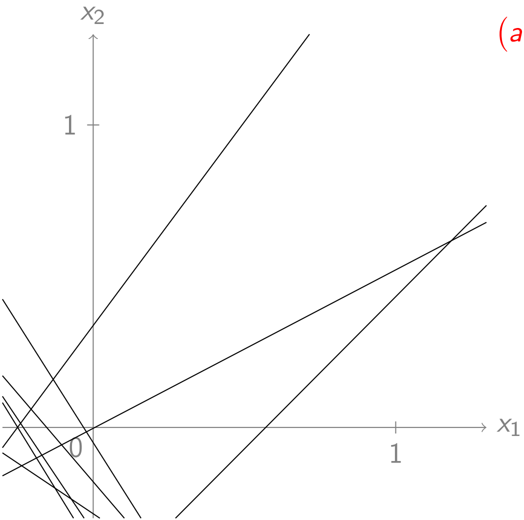

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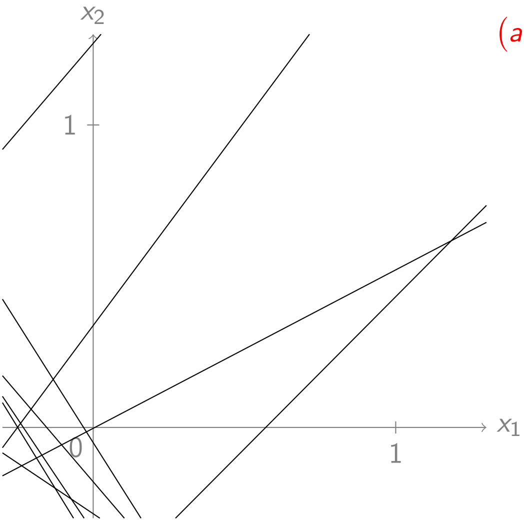

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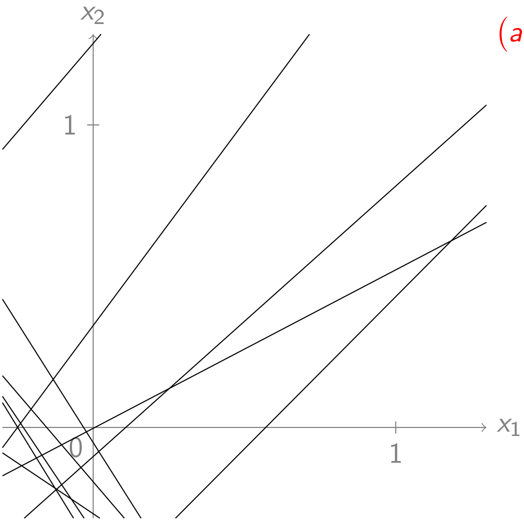

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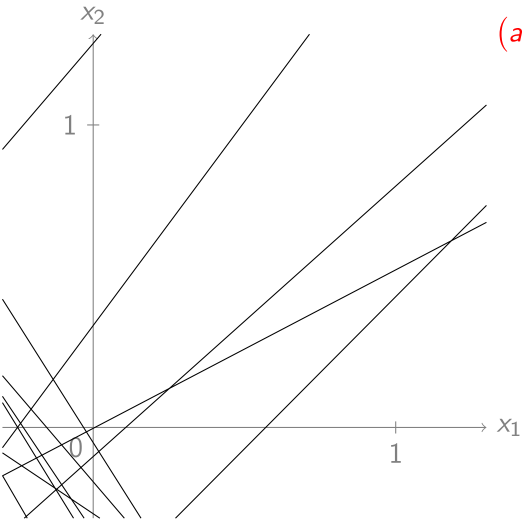

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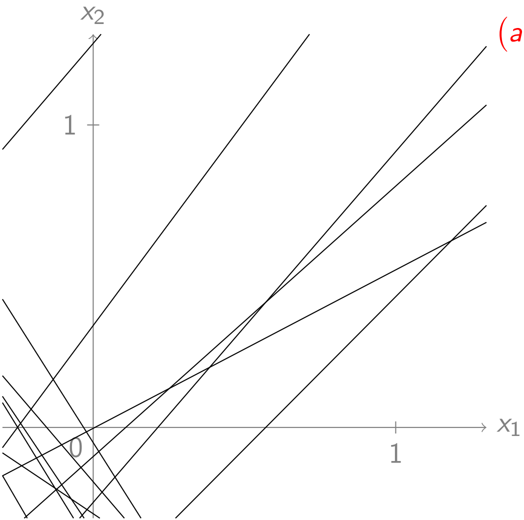

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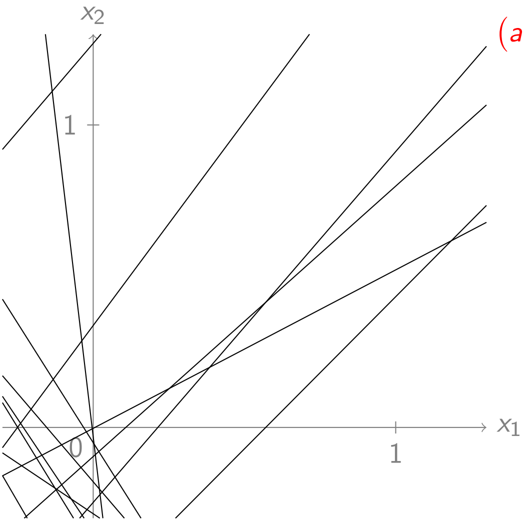

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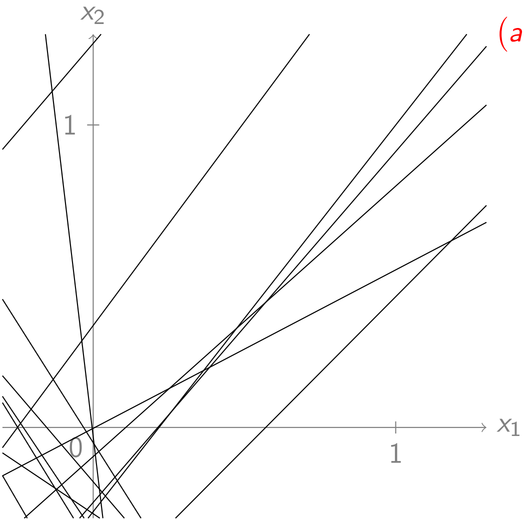

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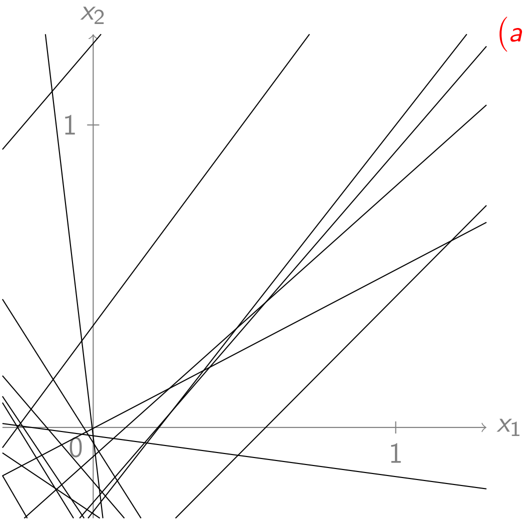

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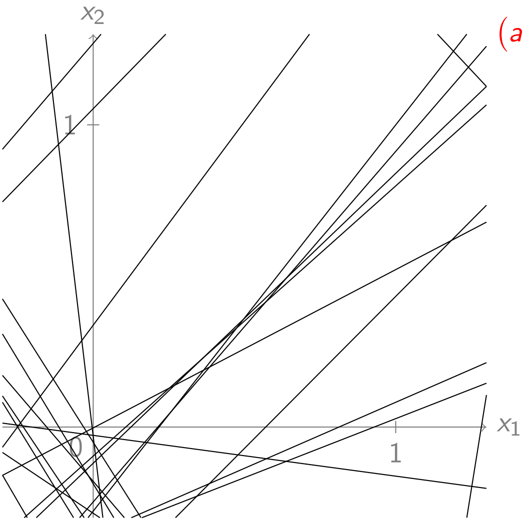

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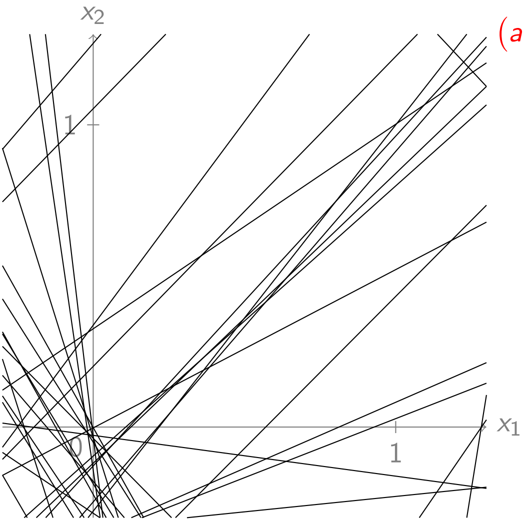

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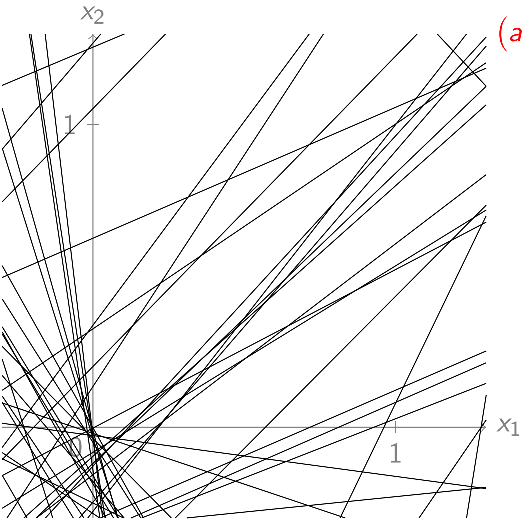

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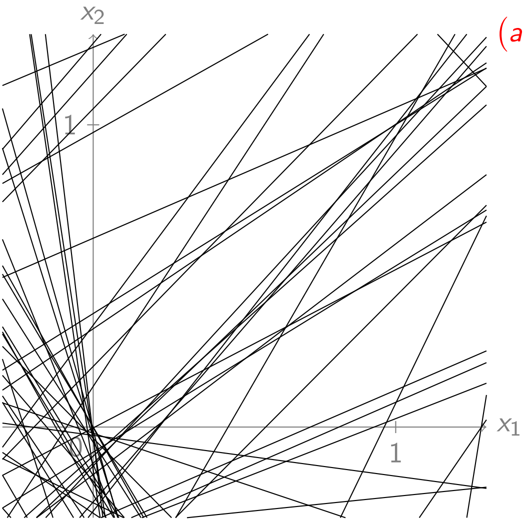

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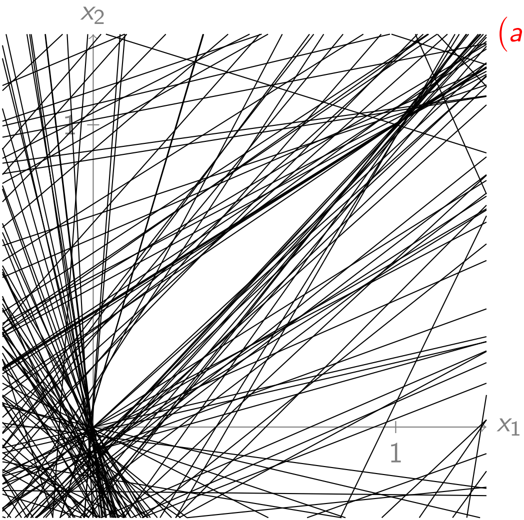

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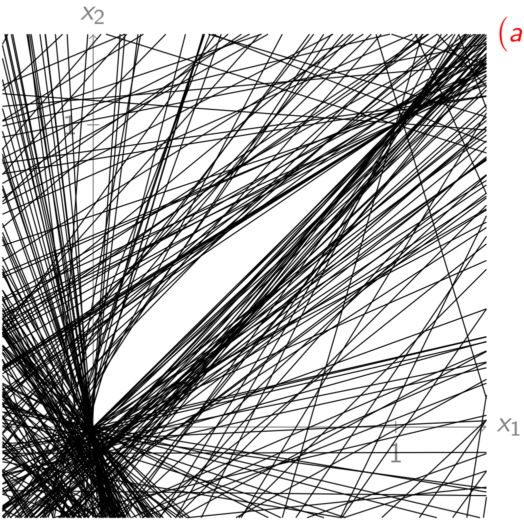

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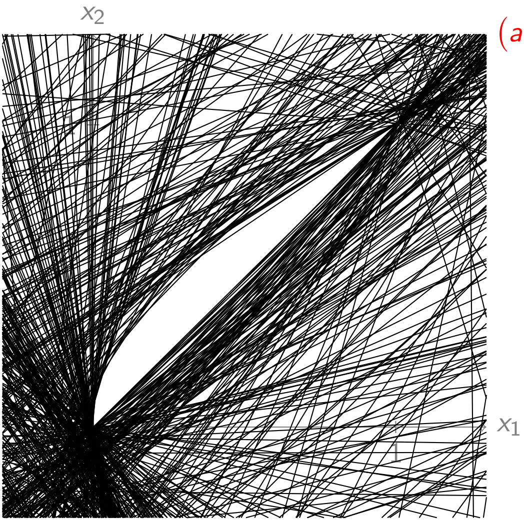

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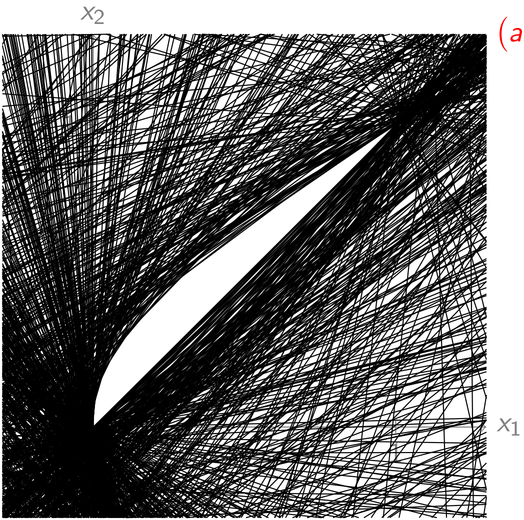

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Describing convex semialgebraic sets by LMIs

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Spectrahedra seem to be easy to deal with algorithmically.
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In recent years, results of Helton & Vinnikov as well as Helton & Nie showed that surprisingly many convex semialgebraic sets are spectrahedra or projections of spectrahedra.

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Let $S \subseteq \mathbb{R}^n$.

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If $\bar{Y}$ is an m-tuple of additional variables, then we call a symmetric linear matrix polynomial $A \in S\mathbb{R}[\bar{X}, \bar{Y}]^{t \times t}$ a semidefinite representation of S if

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Hence S is a spectrahedron if and only if it is LMI representable, and S is a projection of a spectrahedron if and only if it is semidefinitely representable.

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This talk is about

- which sets are spectrahedra and which sets are semidefinitely representable
- how to find LMI representations and semidefinite representations.

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Example. If $S^{(k)} \subseteq \mathbb{R}^n$ is bounded and semidefinitely representable for $k \in \{1, \ldots, \ell\}$, then so is $\mathrm{conv}(\bigcup_{k=1}^{\ell} S^{(k)})$.

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$$\left. \sum_{k=1}^{\ell} \lambda^{(k)} = 1 \quad \wedge \quad x = \sum_{k=1}^{\ell} y^{(k)} \quad \wedge \quad \bigwedge_{k=1}^{\ell} (\lambda^{(k)}, y^{(k)}) \in U^{(k)}\right\}$$

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All these notions are unambigously defined since they do not depend on the chosen basis as the change of bases is given by an invertible linear map.

Describing convex semialgebraic sets by LMIs

This talk is divided into two parts:

Part I. Spectrahedra

Part II. Semidefinitely representable sets

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This will lead us to determinantal representations of polynomials.

Part II. Semidefinitely representable sets

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is empty or an exposed face of S since

$$S \subseteq \{x \in \mathbb{R}^n \mid \langle A(x)u_1, u_1\rangle + \cdots + \langle A(x)u_k, u_k\rangle \geq 0\}.$$

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It remains to show that each face F of $S = \{x \in \mathbb{R}^n \mid A(x) \succeq 0\}$ is of the form $F = \{x \in S \mid U \subseteq \ker A(x)\}$ for a linear subspace of $U \subseteq \mathbb{R}^n$.

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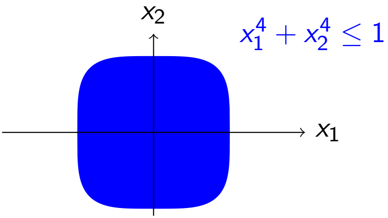

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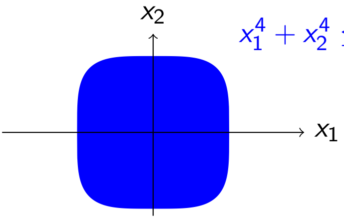

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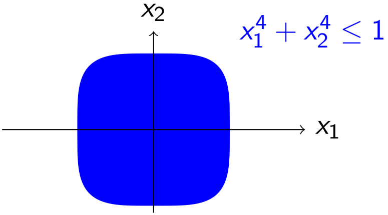

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Thus the assumption that the interior of S is non-empty is not essential and just made for simplicity.

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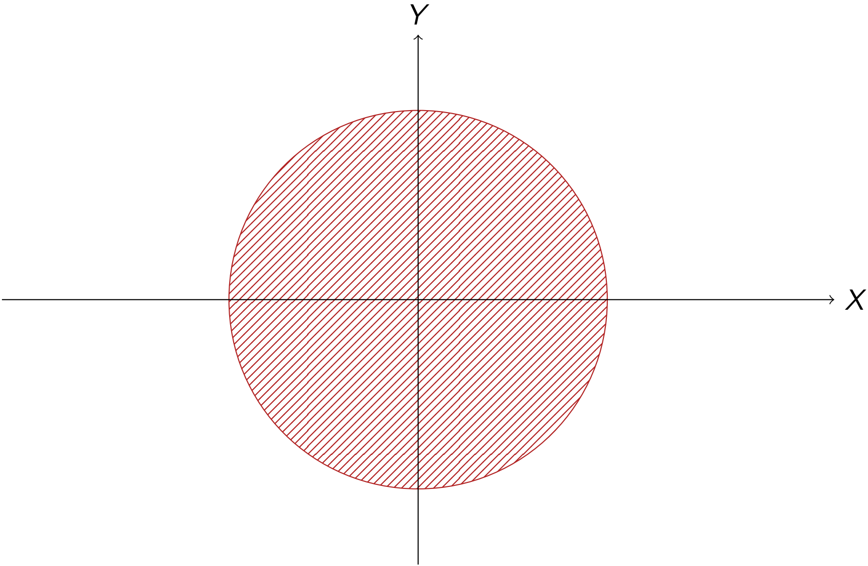
$$S \text{ is an algebraic interior} \quad \& \quad \exists x_0 \in S^\circ \colon \text{min. pol. of } S \text{ is RZ at } x_0$$

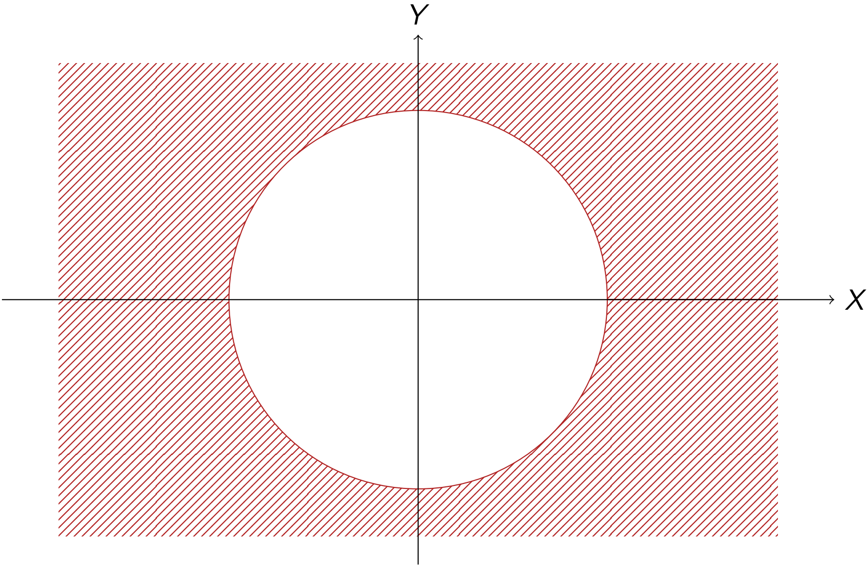
Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $1 - X^2 - Y^2$, rigidly convex



Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $X^2 + Y^2 - 1$, not rigidly convex



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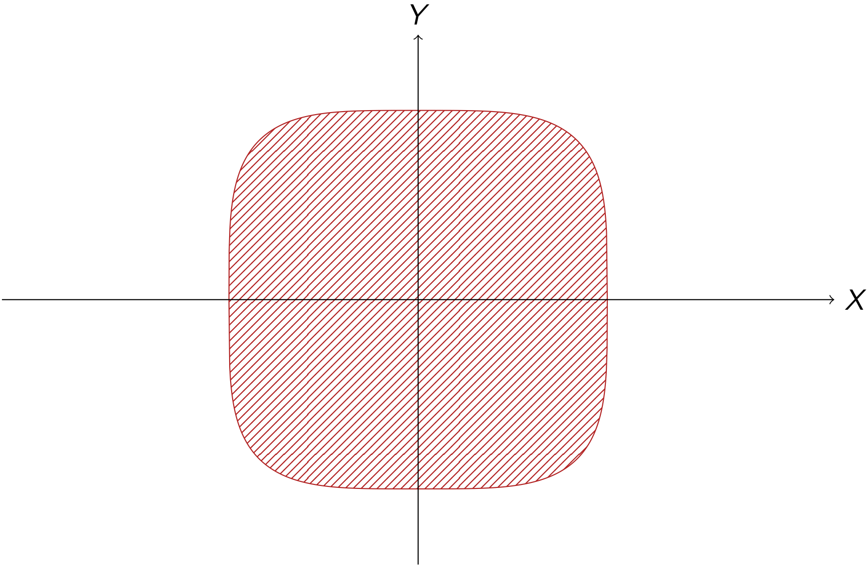
minimal polynomial $1 - X^4 - Y^4$, not rigidly convex



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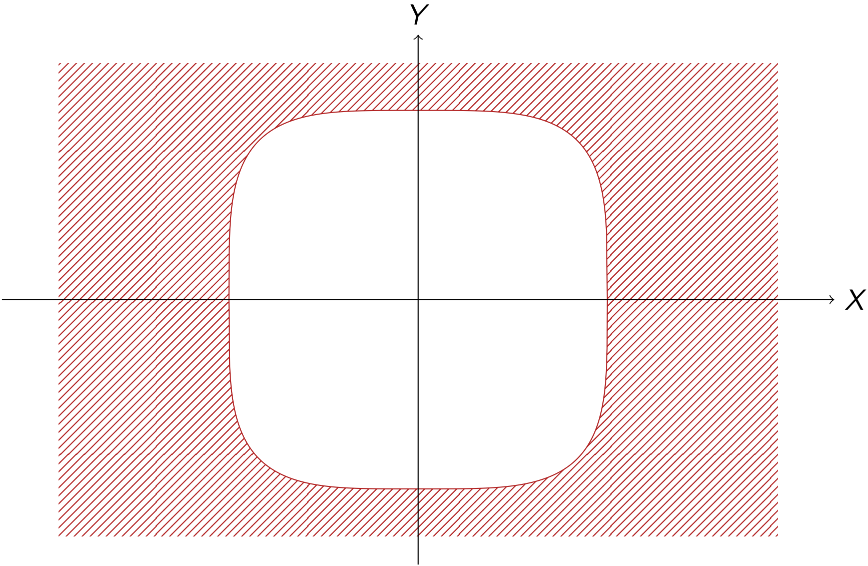

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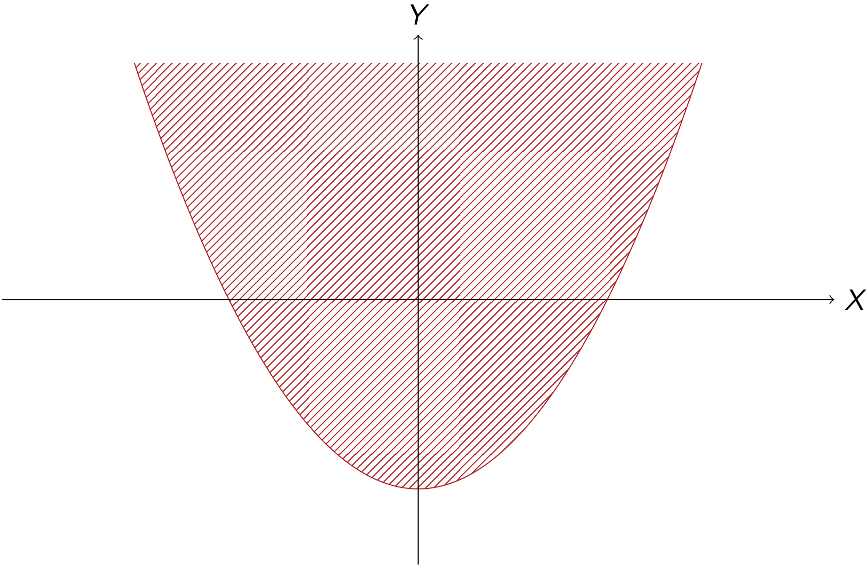
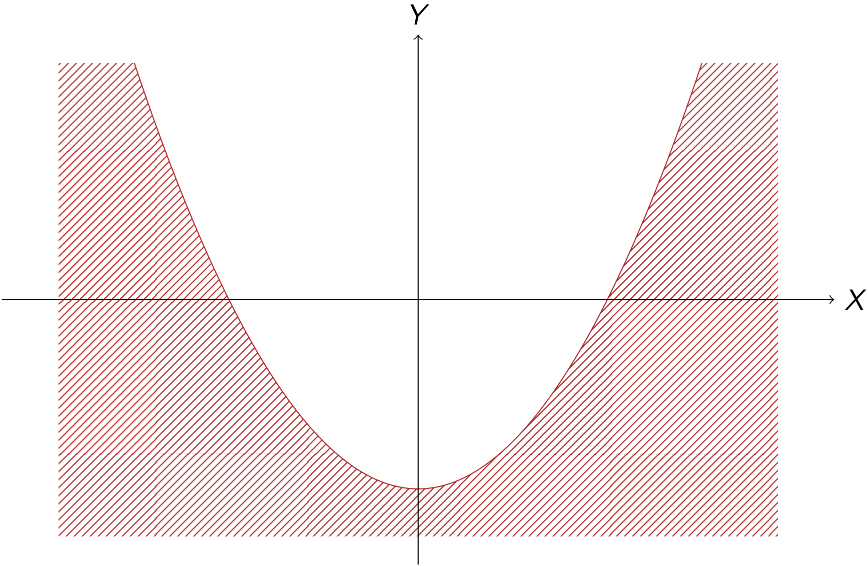
minimal polynomial $Y - X^2 - 1$, rigidly convex



Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $-Y + X^2 + 1$, not rigidly convex



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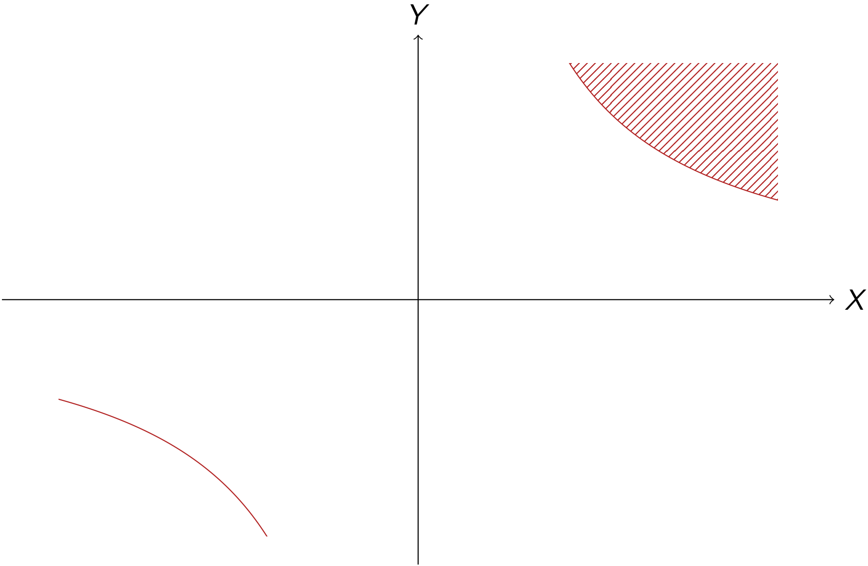

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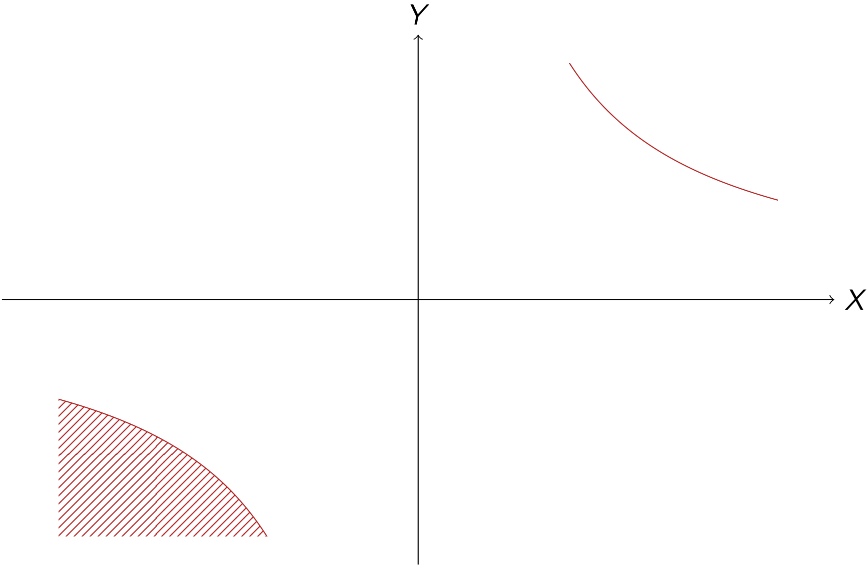
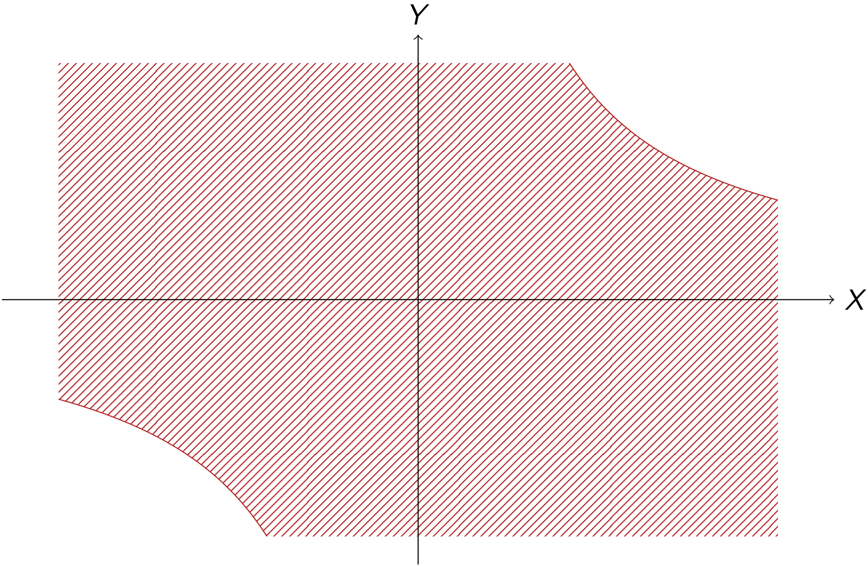


Algebraic interiors, minimal polynomials and rigid convexity

minimal polynomial $1 - XY$, not rigidly convex



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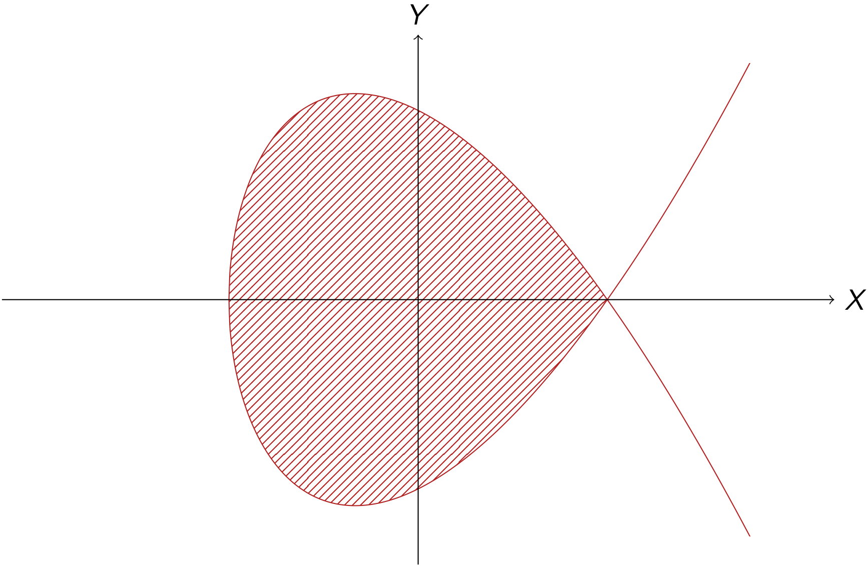

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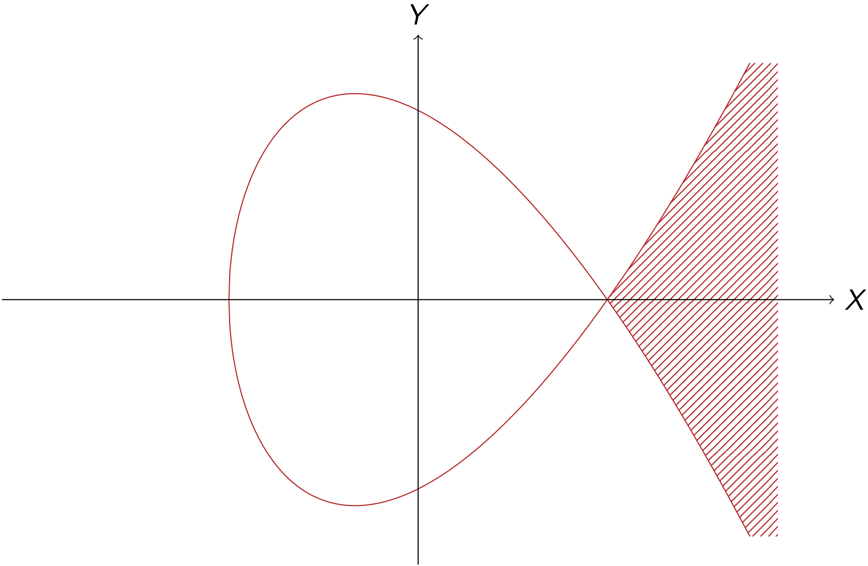

Algebraic interiors, minimal polynomials and rigid convexity

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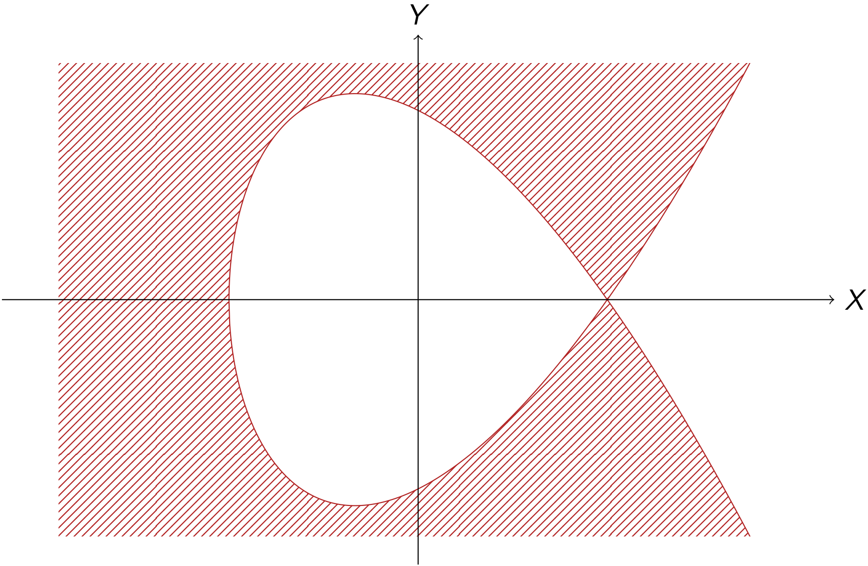

Towards a characterization of spectrahedra

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Conjecture (Helton & Vinnikov 2007).
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New demonstration bypassing polynomials in non-commuting variables and giving an explicit construction: Quarez

Literature on rigid convexity and determinantal representations of real zero polynomials

Helton & Vinnikov: Linear matrix inequality representation of sets
Comm. Pure Appl. Math. 60 (2007), no. 5, 654–674
http://arxiv.org/abs/math.OC/0306180
http://dx.doi.org/10.1002/cpa.20155

Lewis & Parrilo & Ramana: The Lax conjecture is true
Proc. Amer. Math. Soc. 133 (2005), no. 9, 2495–2499
http://arxiv.org/abs/math.OC/0304104
http://dx.doi.org/10.1090/S0002-9939-05-07752-X

Literature on determinantal representations of arbitrary polynomials

Helton & McCullough & Vinnikov: Noncommutative convexity arises from linear matrix inequalities
J. Funct. Anal. 240 (2006), no. 1, 105–191 http://math.ucsd.edu/~helton/osiris/NONCOMMINEQ/convRat.ps
http://dx.doi.org/10.1016/j.jfa.2006.03.018

Quarez: Symmetric determinantal representation of polynomials
http://hal.archives-ouvertes.fr/hal-00275615/fr/

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If $p \in \mathbb{R}[X]$ is a real zero polynomial, i.e., $p(0) > 0$ and $p = \prod_{i=1}^{d} c(X - a_i)$ for some $a_i, c \in \mathbb{R}$, then

$$p = p(0) \prod_{i=1}^{d} (1 - \frac{1}{a_i} X) = p(0) \det \left(I_d - X \operatorname{Diag}\left(\frac{1}{a_1}, \ldots, \frac{1}{a_d} \right) \right).$$

Effective determinantal representations in one variable

Given a polynomial $p \in \mathbb{Q}[X]$ of degree $d = r + 2s$ with at least r real zeros (counted with multiplicity), Quarez constructs by symbolic computation $A \in S\mathbb{Q}^{d \times d}$ such that $p = \det(J + XA)$ where $J = \mathrm{Diag}(\underbrace{1, \ldots, 1}_{r \text{ times}}, \underbrace{1, -1, \ldots, 1, -1}_{s \text{ times}})$.

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Theorem (Quarez). If $p \in \mathbb{R}[X]$ is of degree $d = r + 2s$ with $p(0) \neq 0$. Then p possesses at least r real zeros if and only if there is $A \in S\mathbb{R}^{d\times d}$ such that $p = \det(J + XA)$ with $J = \mathrm{Diag}(\underbrace{1,\dots,1}_{r \text{ times}}, \underbrace{1,-1,\dots,1,-1}_{s \text{ times}})$.

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Quarez: Sturm and Sylvester algorithms revisited via tridiagonal determinantal representations
http://hal.archives-ouvertes.fr/hal-00338925/fr/

Quarez: Représentations déterminantales effectives des polynômes univariés par les matrices flèches
http://hal.archives-ouvertes.fr/hal-00318578/fr/

Definition. Let $p \in \mathbb{R}[\bar{X}]$ be a real zero polynomial of degree d. Then we call

$$R^k p := \frac{\partial^k}{\partial X_0^k} X_0^d \; p\left(\frac{\bar{X}}{X_0}\right)\Bigg|_{X_0=1}$$

the k-th Renegar derivative of p.

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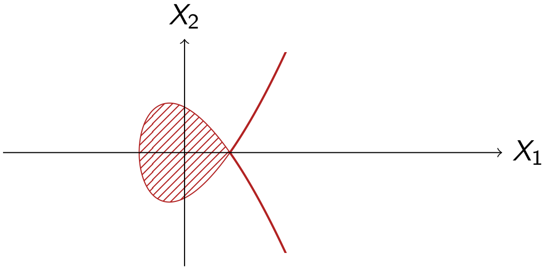

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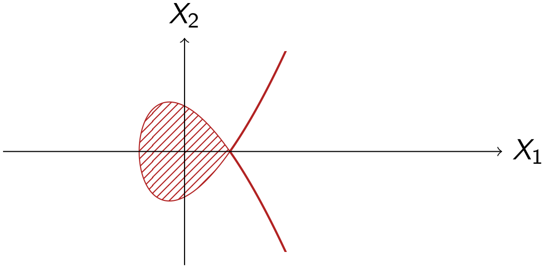

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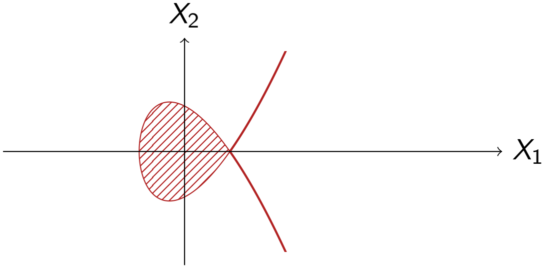

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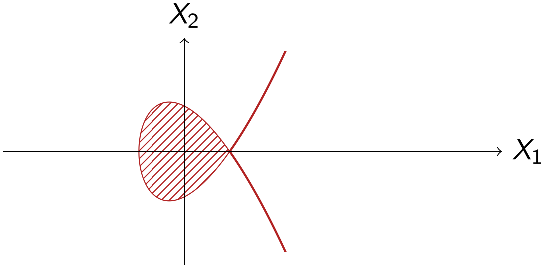

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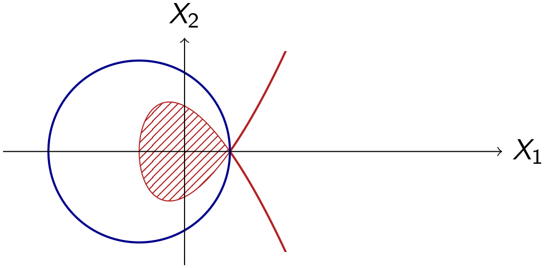

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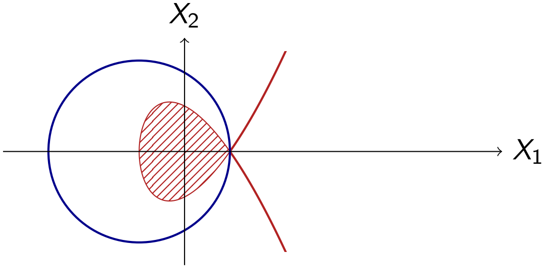

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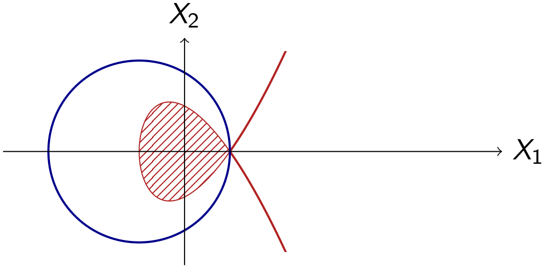

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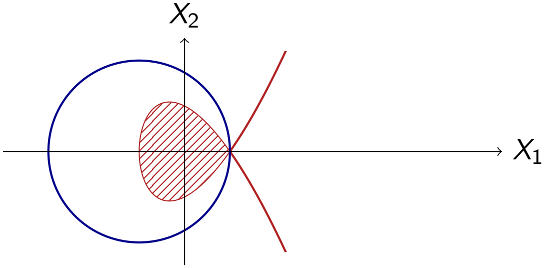

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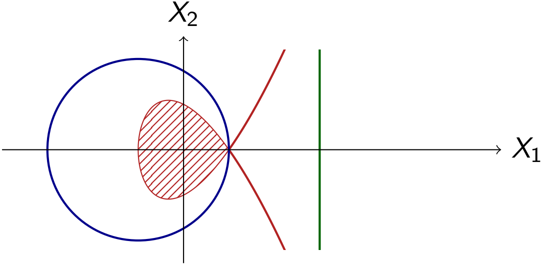

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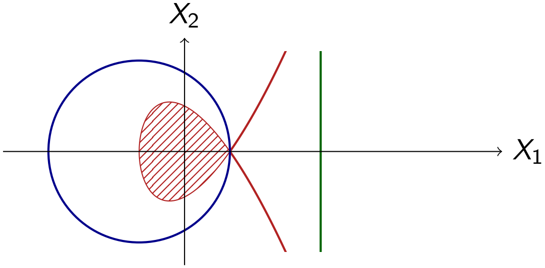
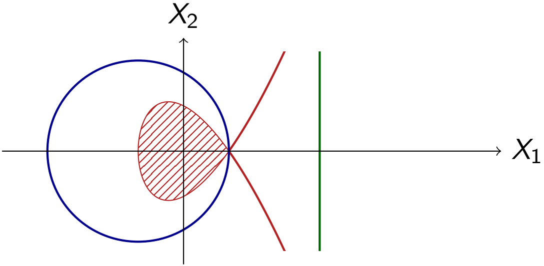


Theorem (Renegar 2006). Let $S \subseteq \mathbb{R}^n$ be rigidly convex with $0 \in S^\circ$ and minimal polynomial p of degree d. Then each $R^k p$ ($k \in \{0, \ldots, d-1\}$) is a real zero polynomial,



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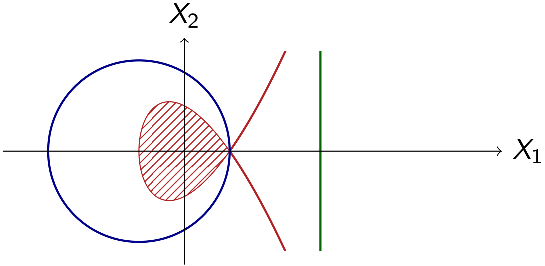

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Moreover, S is basic closed and has only exposed faces. More precisely,

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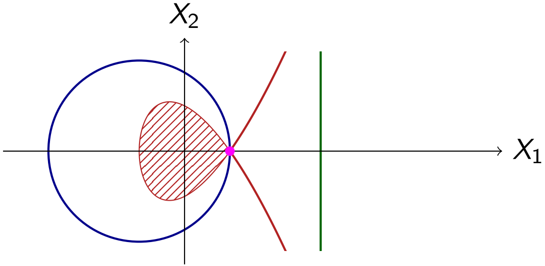

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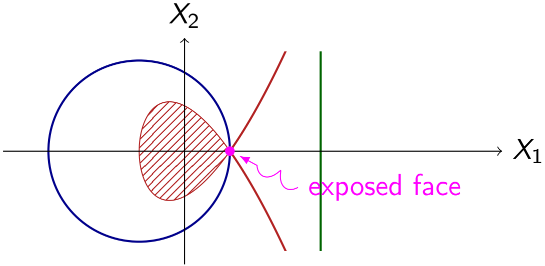
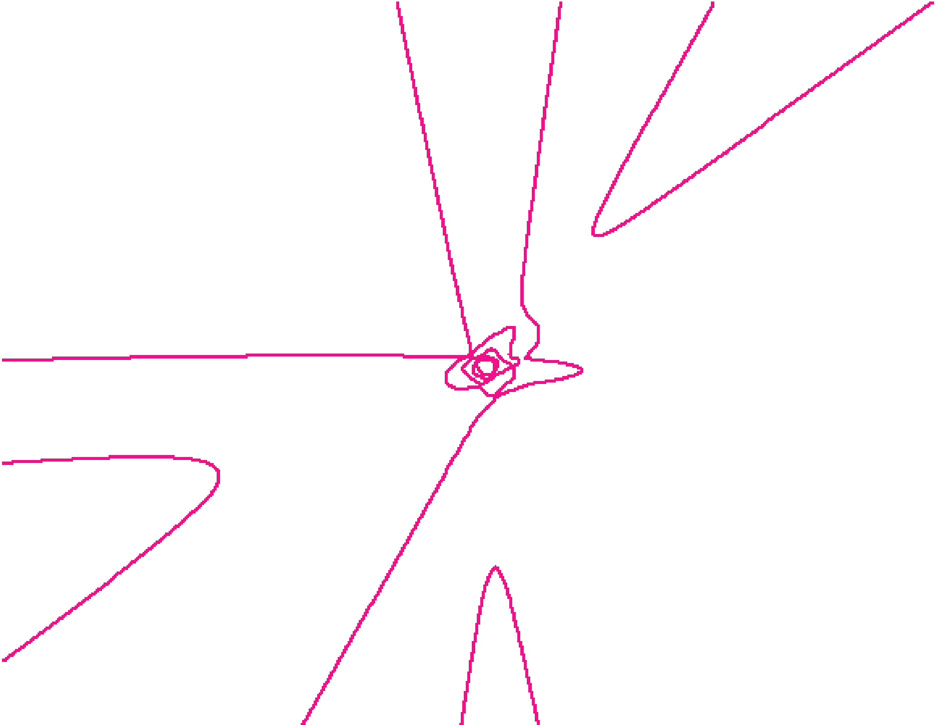
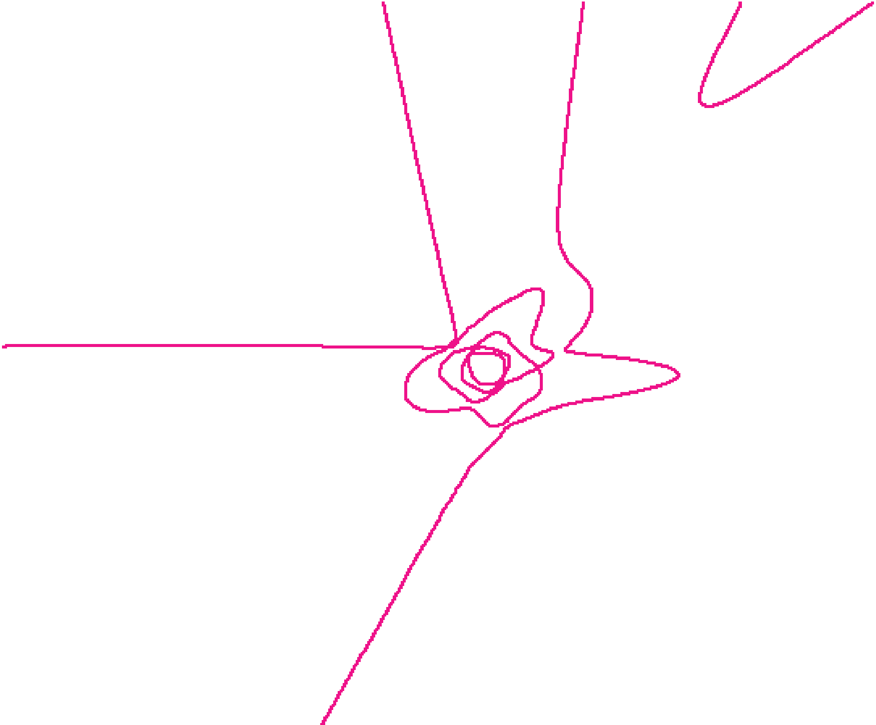
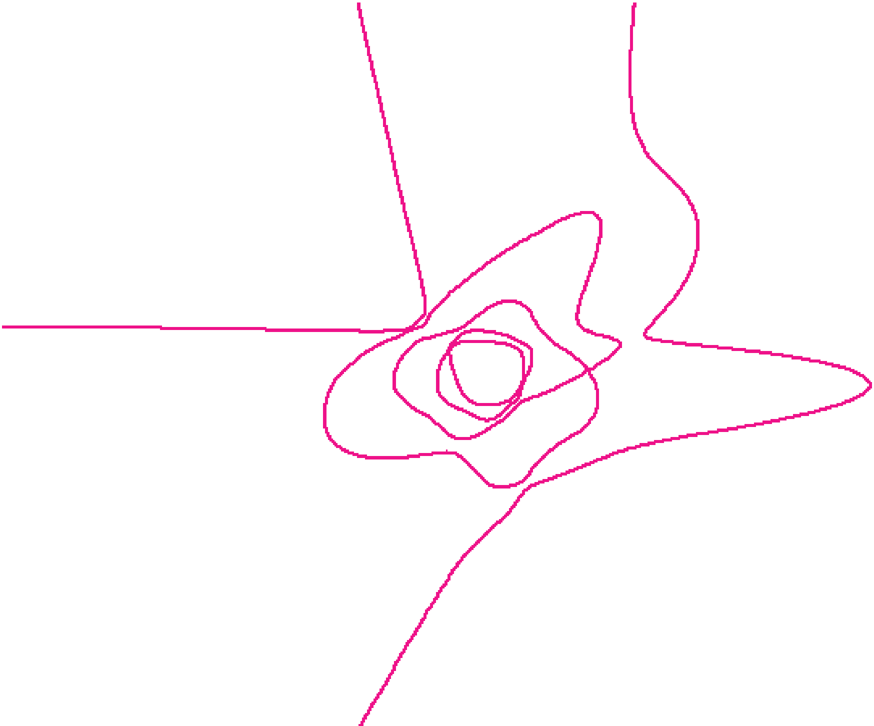
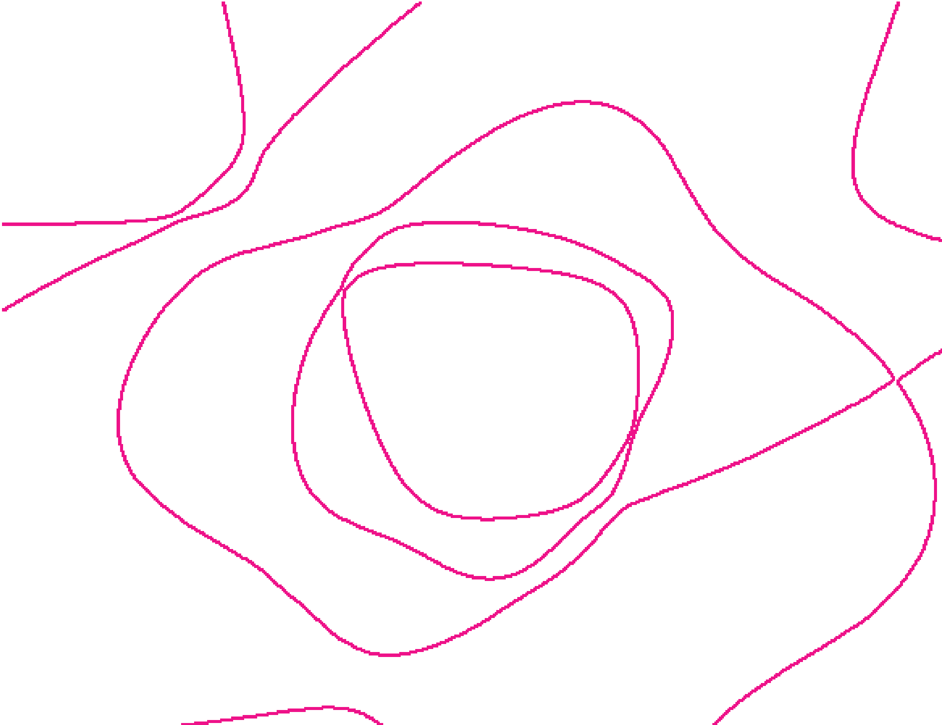
and for $x \in \partial S$ and $k \in \{0, \dots, d-1\}$ maximal such that $x \in \partial S^{(k)}$, there is a unique supporting hyperplane of $S^{(k)}$ at x, and this hyperplane exposes the face in whose relative interior lies x.

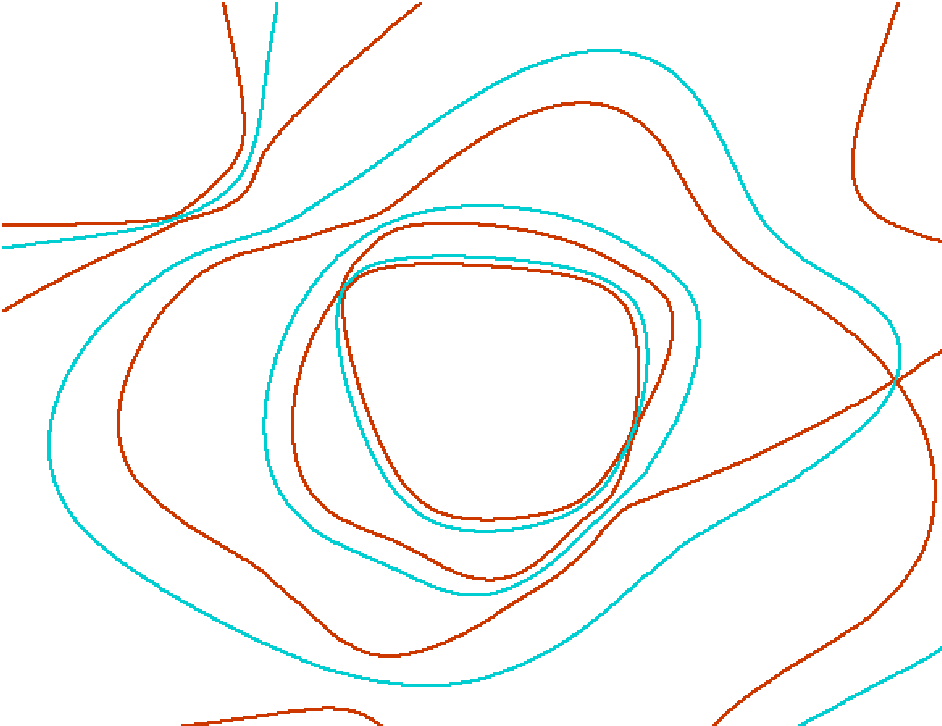


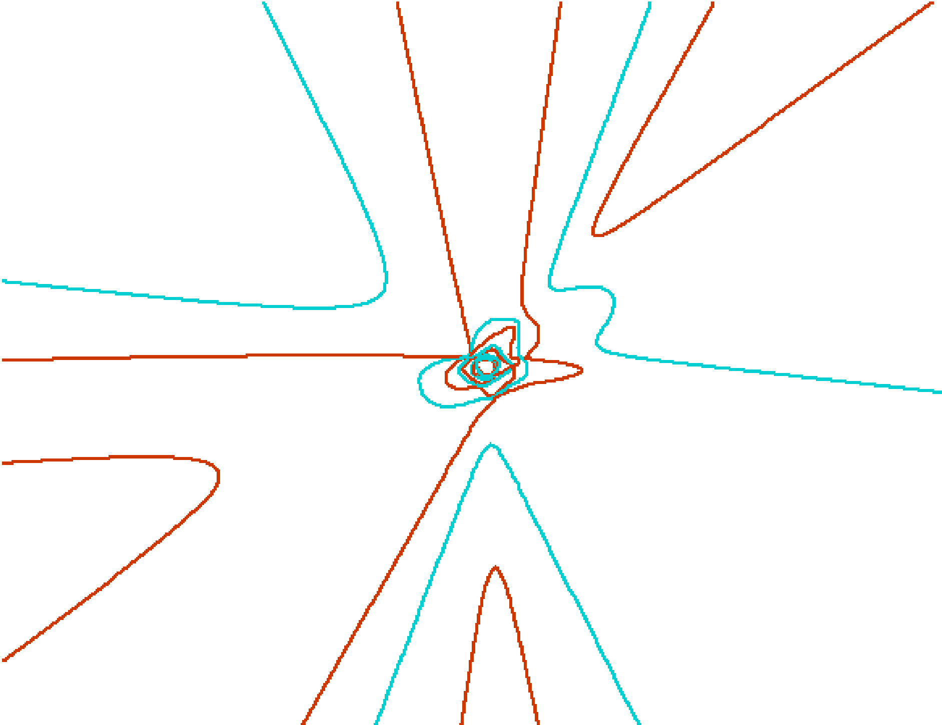


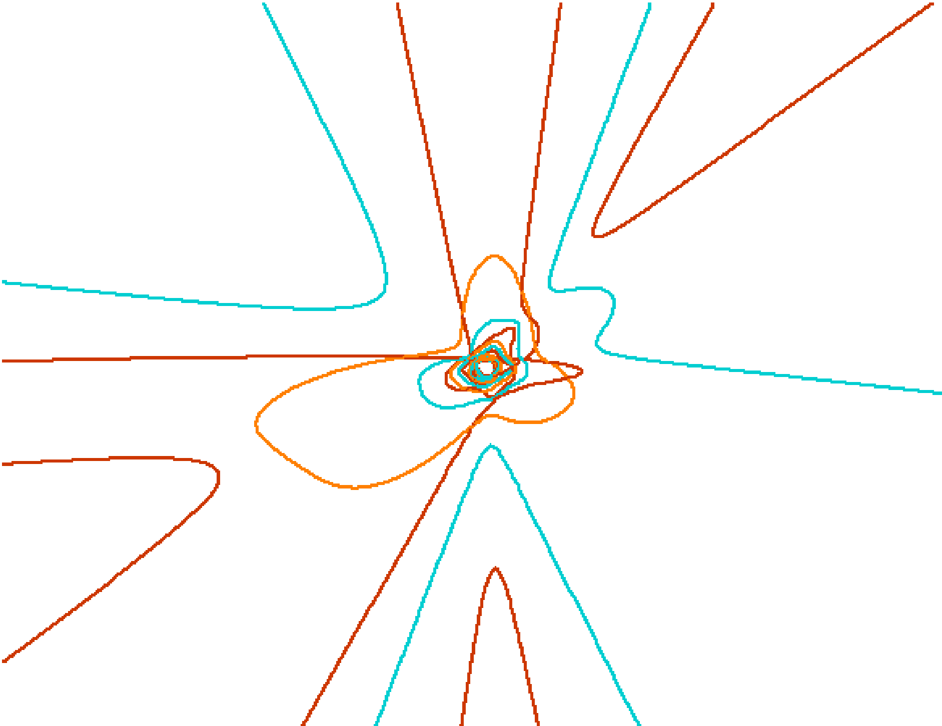


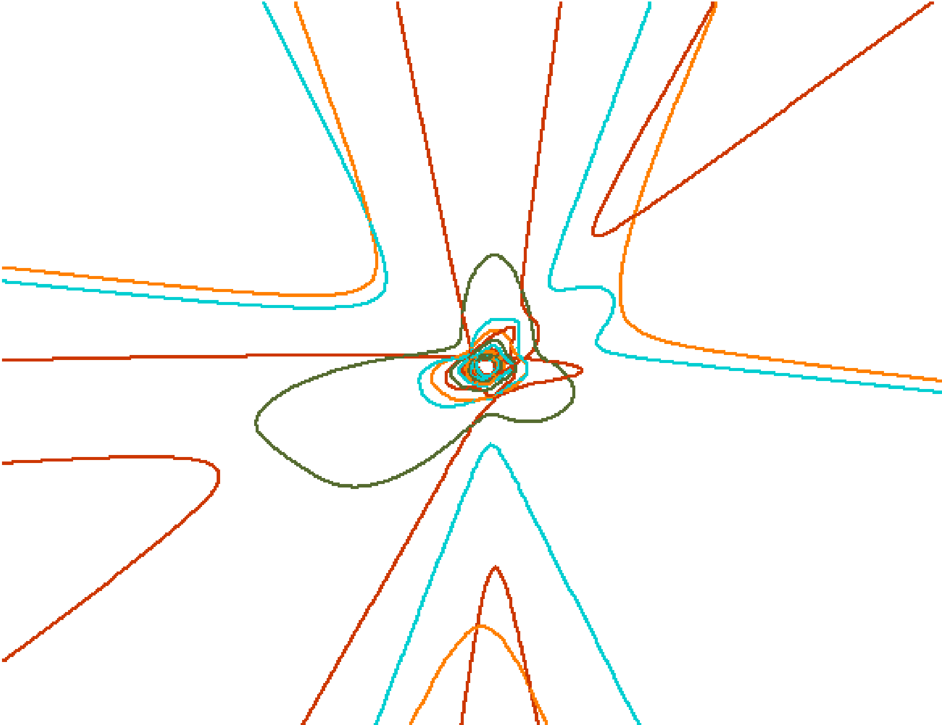


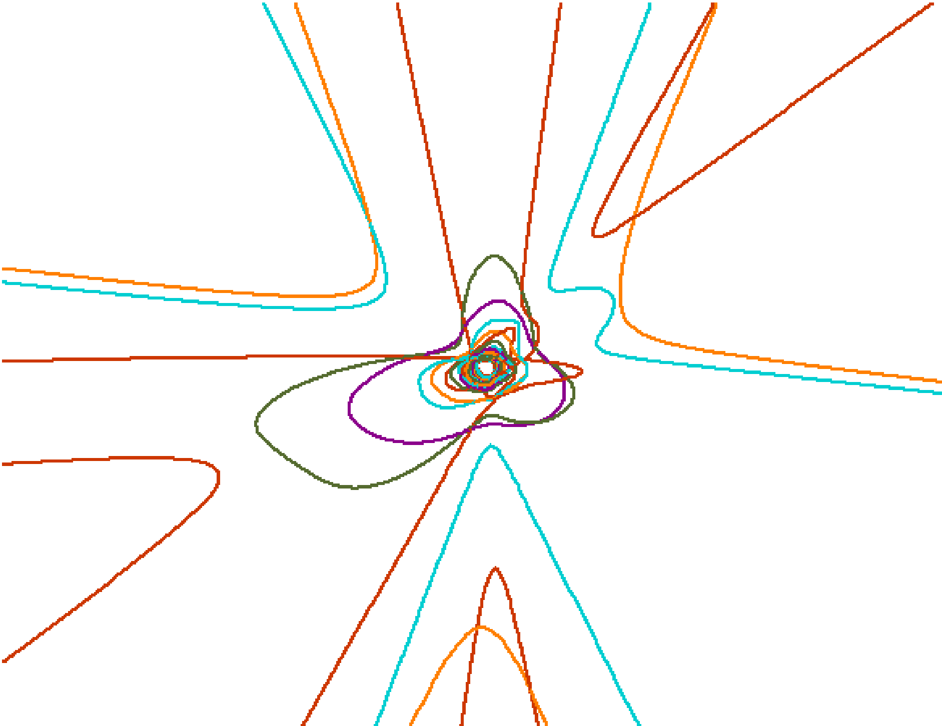


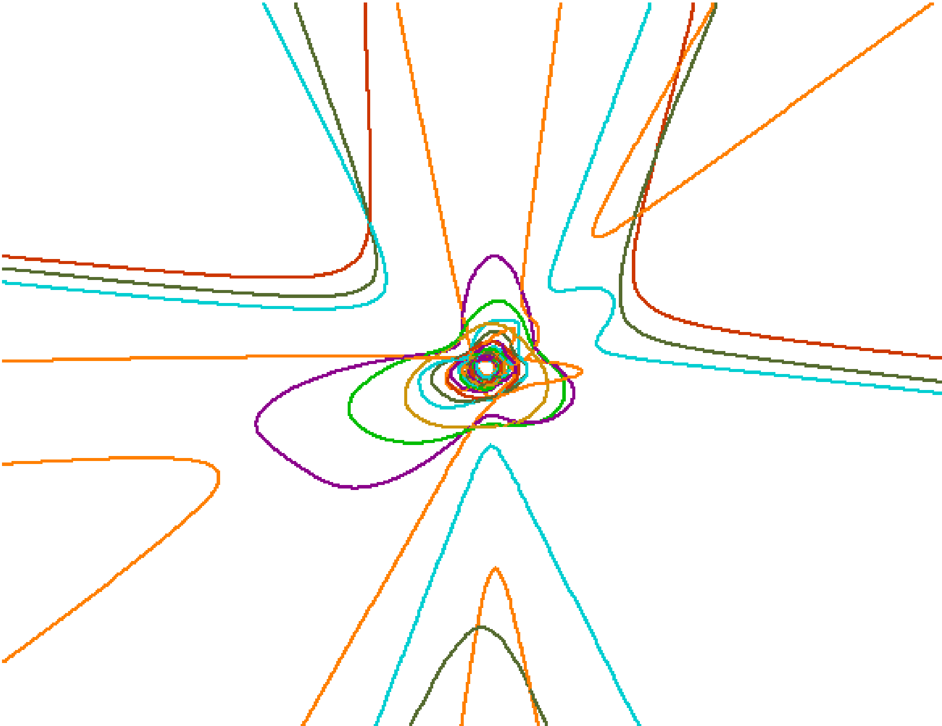


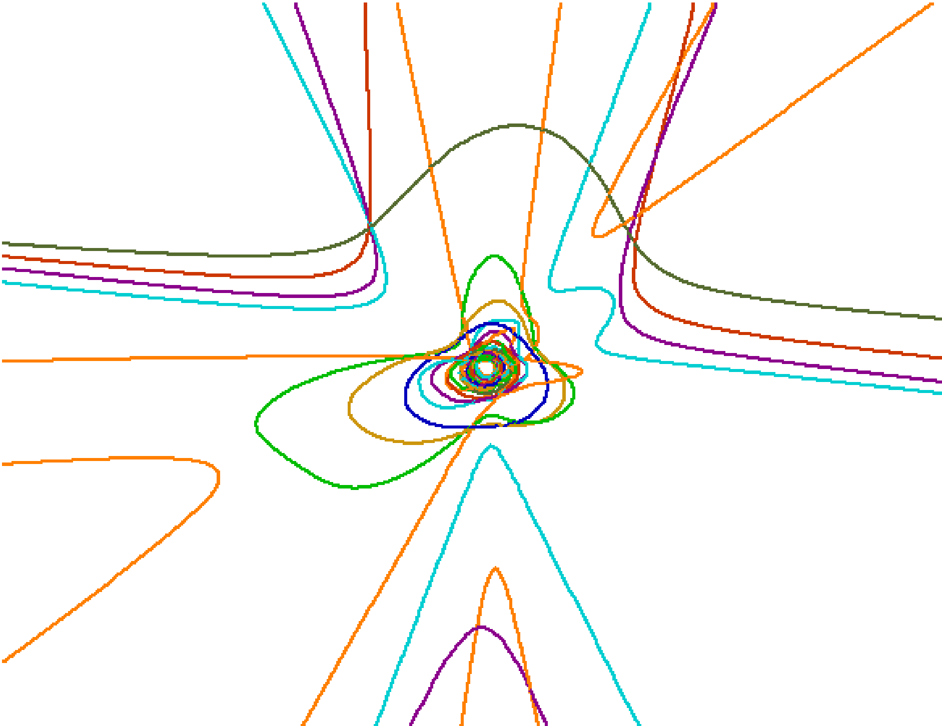


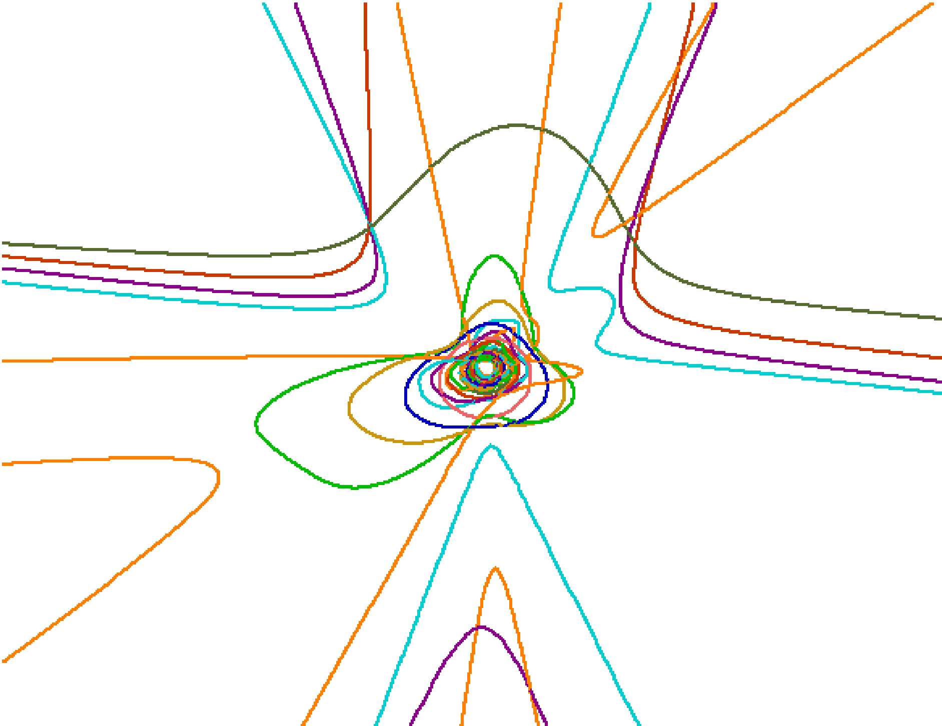


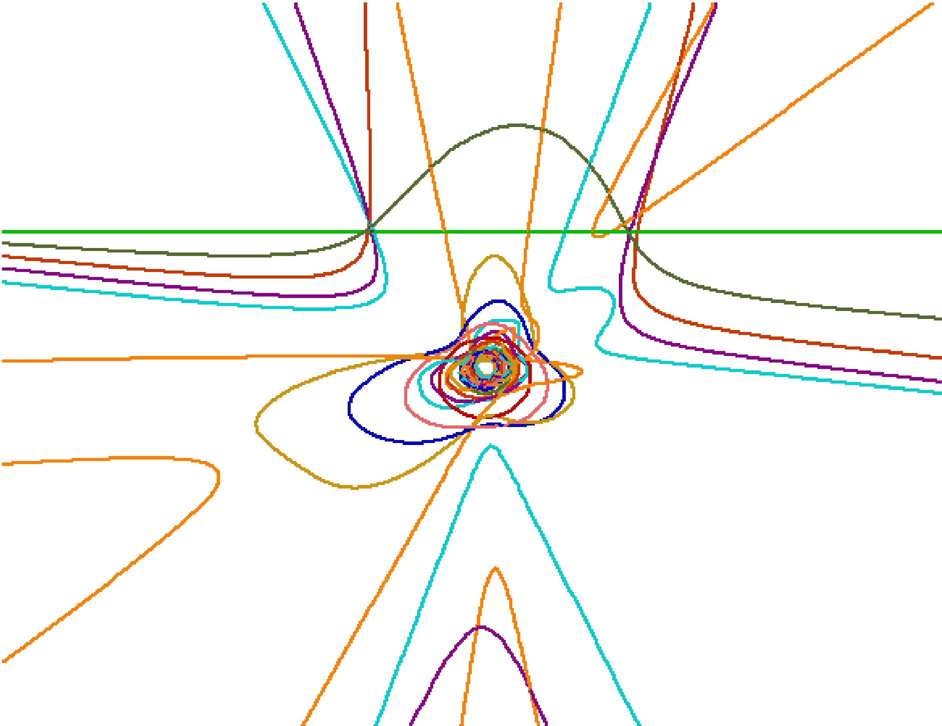












Renegar: Hyperbolic programs, and their derivative relaxations

Found. Comput. Math. 6 (2006), no. 1, 59–79

http://homepage.mac.com/renegar/hyper_progs.pdf

http://dx.doi.org/10.1007/s10208-004-0136-z

Example on the proven Lax conjecture

We have seen geometrically (see below) that

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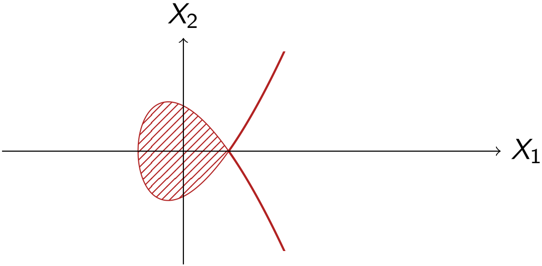

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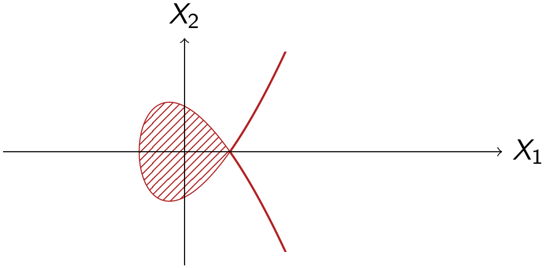

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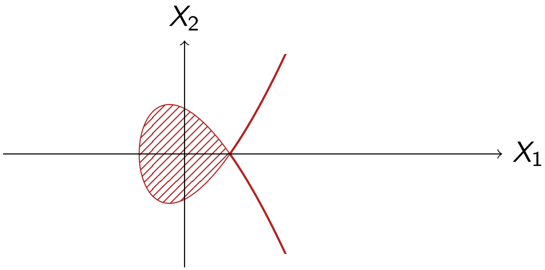

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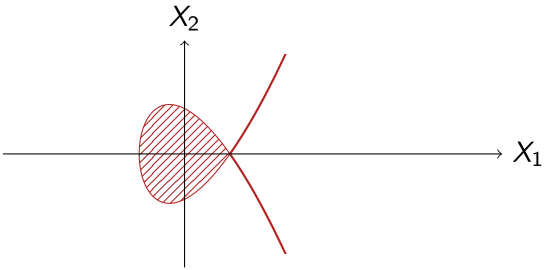
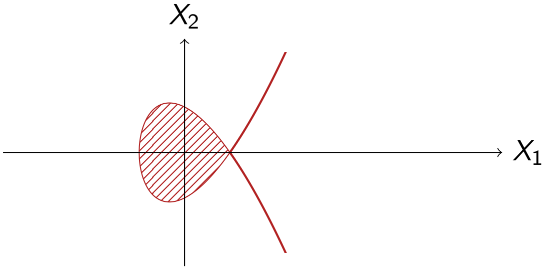


Example on the realization as a basic closed set

Let again $p = \det A$ with $A = \begin{pmatrix} 2-2X_1 & X_2 & 1-X_1 \\ X_2 & 1-X_1 & 0 \\ 1-X_1 & 0 & 1 \end{pmatrix}$.



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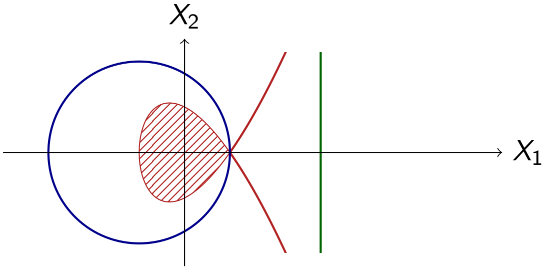

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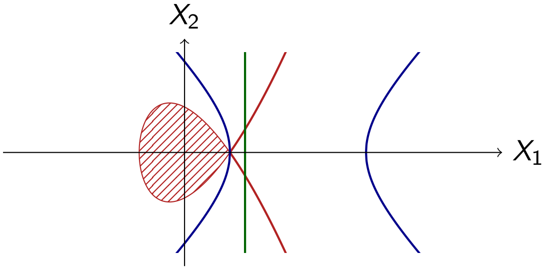
Another way of doing this is to calculate $\det(A + TI_3) = T^3 + (4 - 3X_1)T^2 + (X_1^2 - 5X_1 - X_2^2 + 4)T + p$ and write $S = \{x \in \mathbb{R}^2 \mid p(x) \geq 0, x_1^2 - 5x_1 - x_2^2 + 4 \geq 0, 4 - 3x_1 \geq 0\}$.



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Even this is not known.

Part II. Semidefinitely representable sets

Projections of spectrahedrons

Recall: If $S = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^k \colon A(x, y) \succeq 0\}$ for some symmetric linear matrix polynomial $A \in \mathbb{R}[\bar{X}, \bar{Y}]^{t \times t}$,

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Second big question of the talk:

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Is every convex semialgebraic set semidefinitely representable?

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Example. We have seen that $S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^4 + x_2^4 \leq 1\}$ is not a spectrahedron. However, it is semidefinitely representable since

$$\begin{aligned} S &= \{(x_1, x_2) \in \mathbb{R}^2 \mid \exists y_1, y_2 \in \mathbb{R} : \\ &\qquad 1 - y_1^2 - y_2^2 \geq 0 \quad \& \quad y_1 \geq x_1^2 \quad \& \quad y_2 \geq x_2^2\} \\ &= \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid \exists y_1, y_2 \in \mathbb{R} : \\ &\qquad \begin{pmatrix} 1+y_1 & y_2 \\ y_2 & 1-y_1 \end{pmatrix} \succeq 0 \quad \& \quad \begin{pmatrix} y_1 & x_1 \\ x_1 & 1 \end{pmatrix} \succeq 0 \quad \& \quad \begin{pmatrix} y_2 & x_2 \\ x_2 & 1 \end{pmatrix} \succeq 0 \Big\}. \end{aligned}$$

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The proof of Netzer is constructive and gives rise to simple explicit constructions which preserve for example rational coefficients in the semidefinite representation.

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Each of the methods is scattered over both of the following papers.

How to find semidefinite representations

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to appear in Math. Prog.
http://arxiv.org/abs/0705.4068
http://dx.doi.org/10.1007/s10107-008-0240-y

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How to find semidefinite representations

The basic idea is to use the Lasserre moment relaxation of a basic closed semialgebraic set, or more precisely of a finite system of non-strict polynomial inequalities. We will explain this now.

Lasserre: Convex sets with semidefinite representation
Math. Prog. 120, no. 2 (2009), 457–477
http://hal.archives-ouvertes.fr/docs/00/33/16/65/PDF/SDR-final.pdf http://dx.doi.org/10.1007/s10107-008-0222-0

System of polynomial inequalities

$$\begin{array}{rrrrrrrrrrrrr} & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\ - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\ & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \end{array}$$

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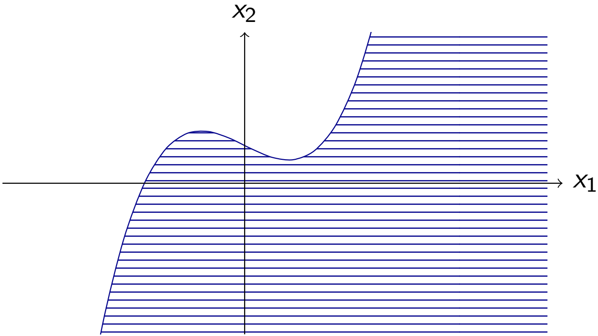

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\end{array}
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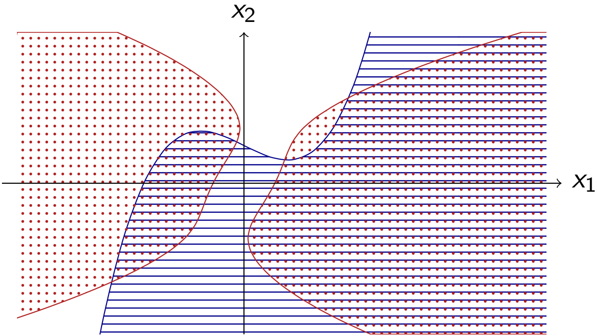

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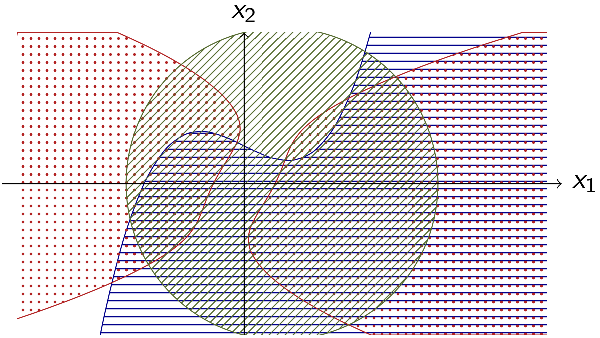

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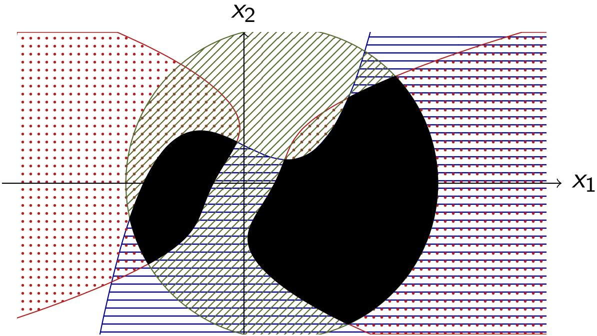

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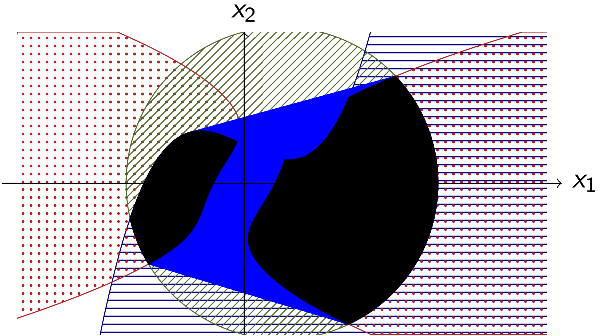

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conv S

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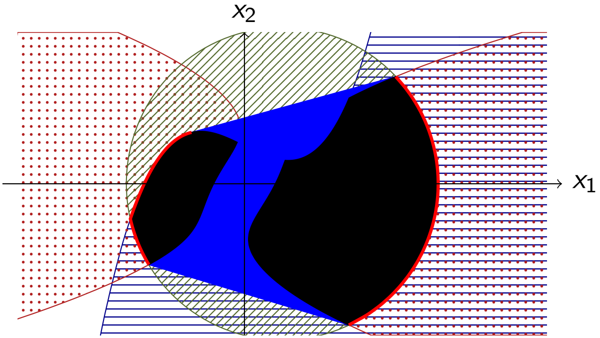

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$S \cap \partial \operatorname{conv} S$

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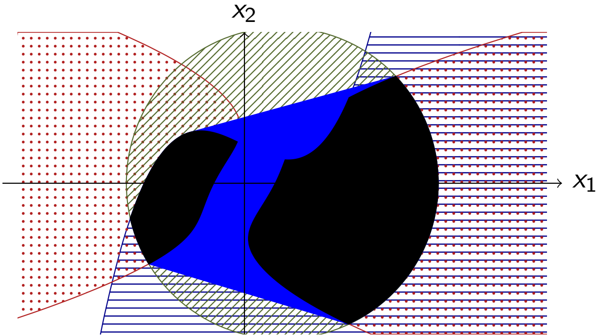

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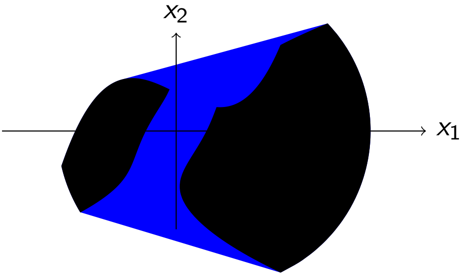

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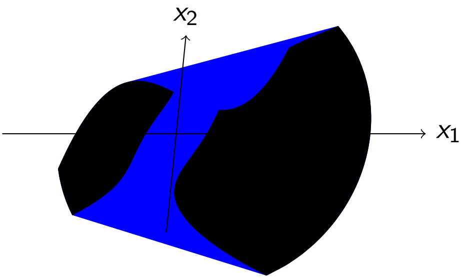

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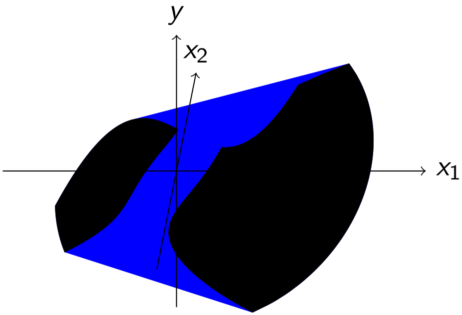

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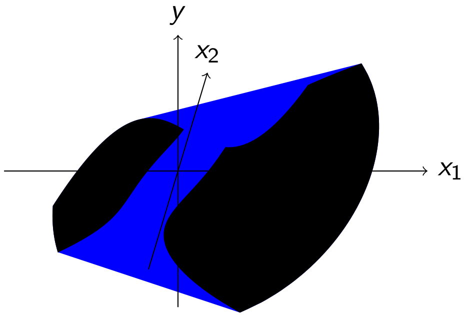
\end{array}
$$

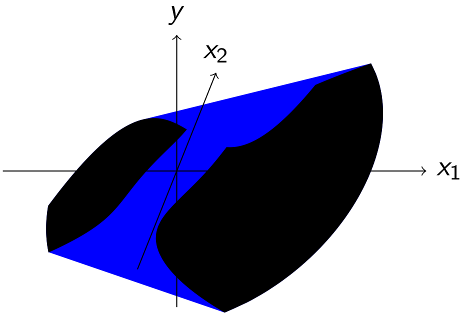


S

conv S

System of polynomial inequalities

$$
\begin{array}{lrcl}
A & -x_1^3 + x_1 + 2x_2 - 1 & \geq & 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$

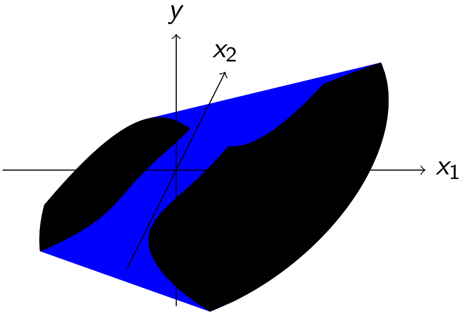


S

conv S

System of polynomial inequalities

$$
\begin{array}{ll}
A & -x_1^3 + x_1 + 2x_2 - 1 \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$



S

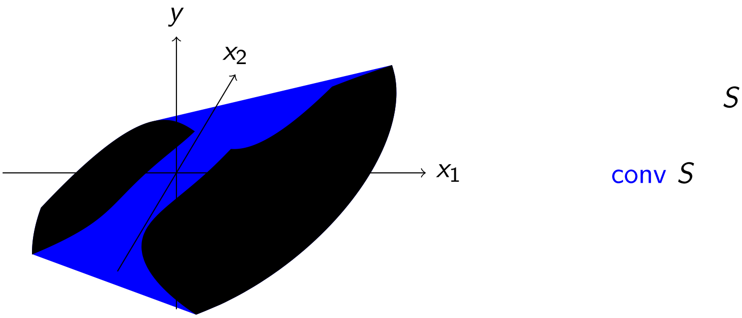
conv S

System of polynomial inequalities

$$A: \quad -x_1^3 + x_1 + 2x_2 - 1 \geq 0$$

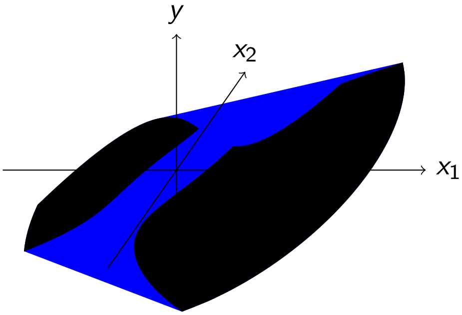
$$B: \quad -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \tfrac{1}{3} \geq 0$$

$$C: \quad -x_1^2 - x_2^2 + x_1 + 4 \geq 0$$



System of polynomial inequalities

$$\begin{array}{llr}
A & & -x_1^3 + x_1 + 2x_2 - 1 \geq 0 \\
B & & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0 \\
C & & -x_1^2 - x_2^2 + x_1 + 4 \geq 0
\end{array}$$

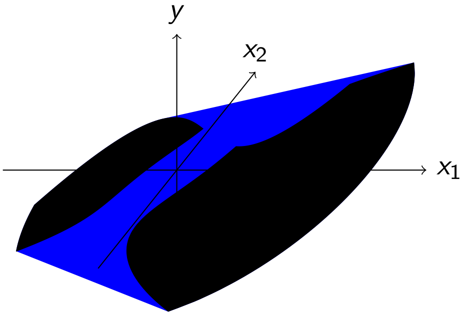


S

conv S

System of polynomial inequalities

A			$-$	x_1^3	$+$	x_1	$+$	$2x_2$	$-$	1	$\geq$	0
B	$-$	x_2^4	$+$	$2x_1^2$	$-$	$2x_1x_2$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	x_1^2	$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0

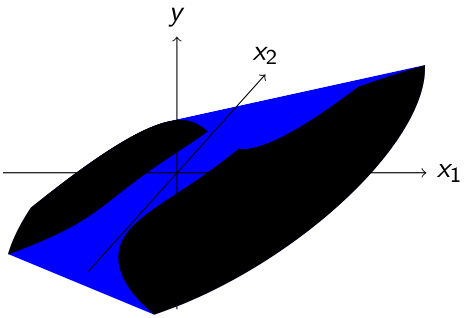


S

conv S

System of polynomial inequalities

$$
\begin{array}{lrrrrrrcl}
A & & -x_1^3 & + x_1 & + 2x_2 & - 1 & \geq & 0 \\
B & -x_2^4 & + 2x_1^2 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq & 0 \\
C & & -x_1^2 & - x_2^2 & + x_1 & + 4 & \geq & 0
\end{array}
$$

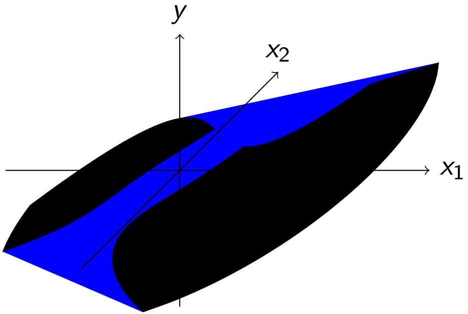


conv S

S

System of polynomial inequalities

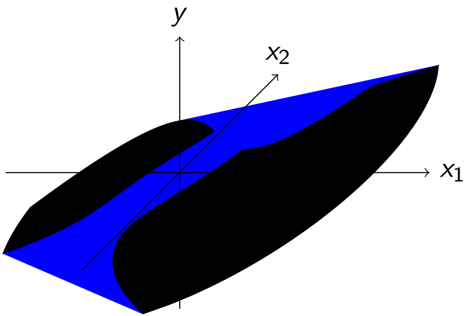
$$\begin{array}{lrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$



conv S

System of polynomial inequalities

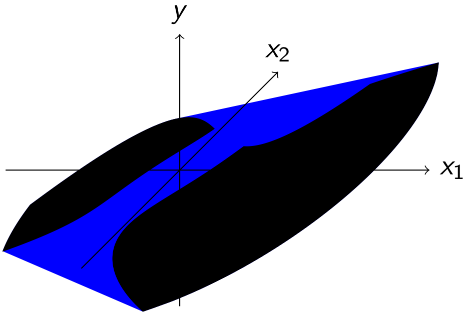
A	$-x_1^3 + x_1 + 2x_2 - 1 \geq 0$
B	$-x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0$
C	$-x_1^2 - x_2^2 + x_1 + 4 \geq 0$



conv S

System of polynomial inequalities

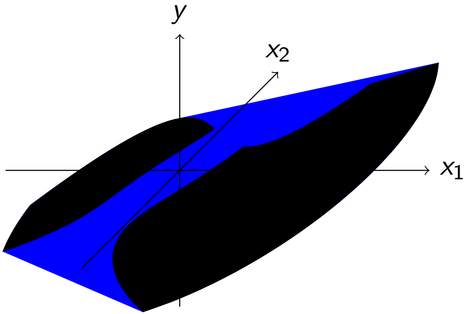
$$
\begin{array}{lrl}
A & -y_1 + x_1 + 2x_2 - 1 & \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq 0
\end{array}
$$



conv S

System of polynomial inequalities

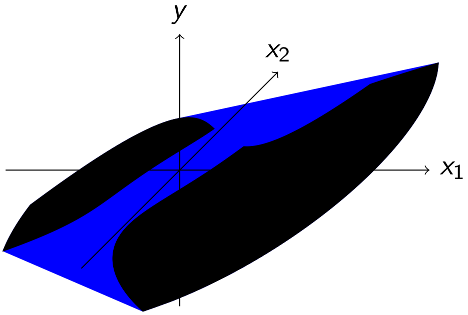
A	$-y_1 + x_1 + 2x_2 - 1 \geq 0$
B	$-x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0$
C	$-x_1^2 - x_2^2 + x_1 + 4 \geq 0$



conv S

System of polynomial inequalities

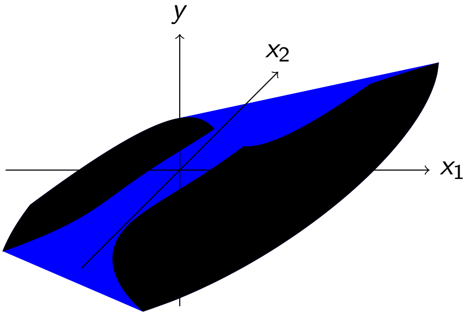
$$
\begin{array}{lrrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

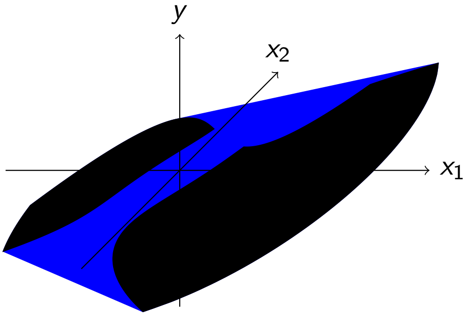
$$
\begin{array}{lrcl}
A & -y_1 + x_1 + 2x_2 - 1 & \geq & 0 \\
B & -y_2 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

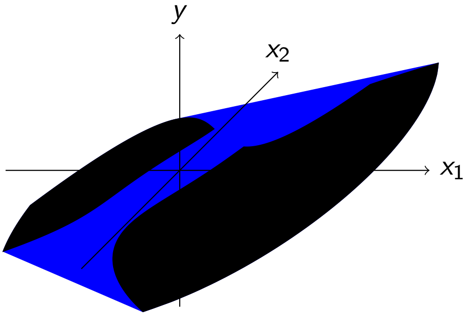
$$
\begin{array}{lrrrrrr}
A & & -y_1 & +x_1 & +2x_2 & -1 & \geq 0 \\
B & -y_2 & +2y_3 & -2x_1x_2 & +x_2^2 & -\frac{1}{3} & \geq 0 \\
C & & -y_3 & -x_2^2 & +x_1 & +4 & \geq 0
\end{array}
$$



conv S

System of polynomial inequalities

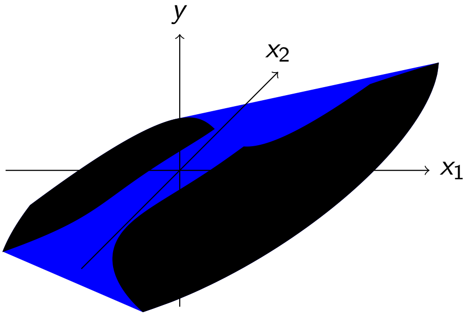
$$
\begin{array}{llr}
A & & - y_1 + x_1 + 2x_2 - 1 \geq 0 \\
B & & - y_2 + 2y_3 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0 \\
C & & - y_3 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$



conv S

System of polynomial inequalities

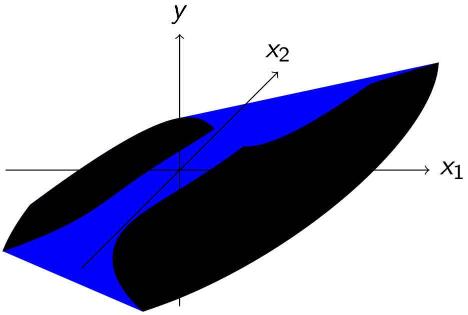
$$
\begin{array}{lrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

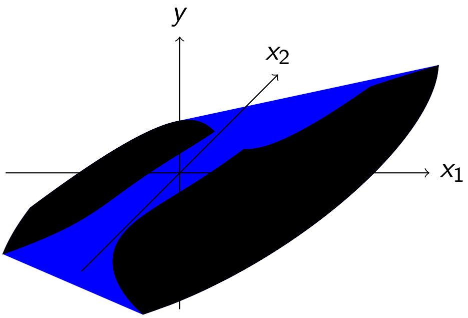
$$
\begin{array}{lrcl}
A & -y_1 + x_1 + 2x_2 - 1 & \geq & 0 \\
B & -y_2 + 2y_3 - 2y_4 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -y_3 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$



conv S

System of linear inequalities

$$\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$



conv S

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$\begin{array}{lrrrrrrl}
A & & -x_1^3 & +x_1 & +2x_2 & -1 & \geq & 0 \\
B & -x_2^4 & +2x_1^2 & -2x_1x_2 & +x_2^2 & -\frac{1}{3} & \geq & 0 \\
C & & -x_1^2 & -x_2^2 & +x_1 & +4 & \geq & 0
\end{array}$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$\begin{array}{lrrrrrl}
A & & -x_1^3 & + x_1 & + 2x_2 & - 1 & \geq 0 \\
B & -x_2^4 & + 2x_1^2 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq 0 \\
C & & -x_1^2 & - x_2^2 & + x_1 & + 4 & \geq 0
\end{array}$$

redundant:

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} & & & & & & & & & & & & \\
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0
\end{array}$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} \\
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0
\end{array}$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} & & & & & & & & & & & & \\
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0
\end{array}$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} & & & & & & & & & & & & \\
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0
\end{array}$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} \\
AB & & x_1^3x_2^4 & - & \ldots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

A			$-$	x_1^3	$+$	x_1	$+$	$2x_2$	$-$	1	$\geq$	0
B	$-$	x_2^4	$+$	$2x_1^2$	$-$	$2x_1x_2$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	x_1^2	$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0
redundant:												
AB		$x_1^3x_2^4$	$-$	$\ldots$	$-$	x_2^2	$-$	$\frac{2}{3}x_2$	$+$	$\frac{1}{3}$	$\geq$	0
AC		x_1^5	$+$	$\ldots$	$-$	x_1	$+$	$8x_2$	$-$	4	$\geq$	0
ABC	$-$	$x_1^5x_2^4$	$+$	$\ldots$	$-$	$\frac{13}{3}x_2^2$	$-$	$\frac{8}{3}x_2$	$+$	$\frac{4}{3}$	$\geq$	0
D^2						x_1^2	$-$	$2x_1x_2$	$+$	x_2^2	$\geq$	0
D^2C	$-$	x_1^4	$+$	$\ldots$	$+$	$4x_1^2$	$+$	$4x_1x_2$	$+$	$4x_2^2$	$\geq$	0

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant:

$$\begin{array}{lrrrrrrrrrrrrrr}
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & x_1^3x_2^4 & - & \ldots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrrr}
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrl}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} & & & & & & & & & & & & \\
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} & & & & & & & & & & & & \\
AB & & x_1^3x_2^4 & - & \ldots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} & & & & & & & & & & & & \\
AB & & x_1^3 x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & x_2^2 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & x_1^3 x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & x_2^2 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrrr}
AB & & x_1^3 x_2^4 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & x_1^3 x_2^4 & - & \ldots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \ldots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrrr}
AB & & y_6 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & & y_6 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & y_6 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{llrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & y_6 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & y_6 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & y_{13} & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} & & & & & & & & & & & & \\
AB & & y_6 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & y_{13} & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

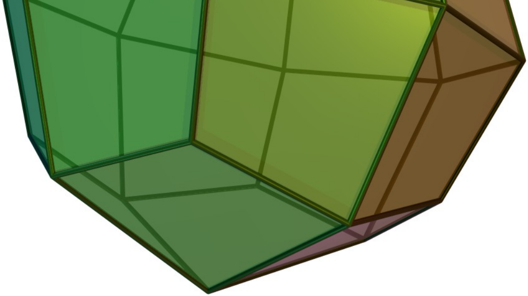
System of polynomial inequalities

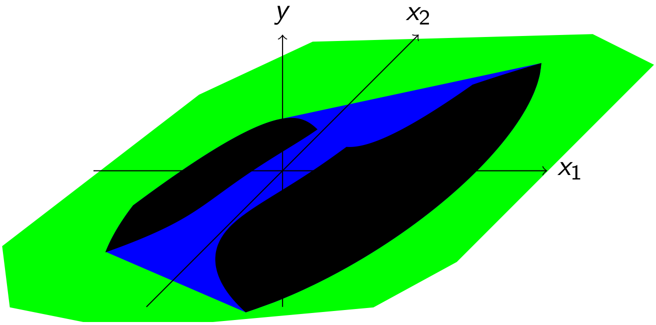
Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & & y_6 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & y_{13} & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & y_{18} & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$





conv S

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0 \qquad \iff$$

$$(a+bx_1+cx_2+dx_1^2+ex_1x_2+fx_2^2)\begin{pmatrix}1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\\e\\f\end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrr}
A & & -x_1^3 & + & x_1 & + 2x_2 & - 1 & \geq & 0 \\
B & -x_2^4 & +2x_1^2 & - & 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq & 0 \\
C & & -x_1^2 & - & x_2^2 & + x_1 & + 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0 \qquad \Longleftrightarrow$$

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrl}
A & & -x_1^3 & +x_1 & +2x_2 & -1 & \geq & 0\\
B & -x_2^4 & +2x_1^2 & -2x_1x_2 & +x_2^2 & -\frac{1}{3} & \geq & 0\\
C & & -x_1^2 & -x_2^2 & +x_1 & +4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & x_1x_2 & x_2^2 \\
x_1 & y_3 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrl}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & x_1x_2 & x_2^2 \\ x_1 & y_3 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & x_2^2 \\ x_1 & y_3 & y_4 & y_1 & x_1^2 x_2 & x_1 x_2^2 \\ x_2 & y_4 & x_2^2 & x_1^2 x_2 & x_1 x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2 x_2 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & x_1^2 x_2 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ x_2^2 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & x_2^2 \\ x_1 & y_3 & y_4 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & y_4 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ y_4 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrr}
A & & & -y_1 & +x_1 & +2x_2 & -1 & \geq & 0 \\
B & -y_2 & +2y_3 & -2y_4 & +y_5 & -\frac{1}{3} & & \geq & 0 \\
C & & & -y_3 & -y_5 & +x_1 & +4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & y_4 & y_5 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ y_4 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ y_5 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & y_4 & y_5 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ y_4 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ y_5 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & x_1x_2^2 \\ x_2 & y_4 & y_5 & y_6 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & y_6 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ y_4 & y_6 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ y_5 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & x_1x_2^2 \\
x_2 & y_4 & y_5 & y_6 & x_1x_2^2 & x_2^3 \\
y_3 & y_1 & y_6 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
y_4 & y_6 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
y_5 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrl}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\
y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\
y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\ y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\ y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{llllllllllllll}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & x_1^2x_2^2 \\
y_4 & y_6 & y_7 & y_{10} & x_1^2x_2^2 & x_1x_2^3 \\
y_5 & y_7 & y_9 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & y_{10} & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\ y_4 & y_6 & y_7 & y_{10} & y_{11} & x_1 x_2^3 \\ y_5 & y_7 & y_9 & y_{11} & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\
y_4 & y_6 & y_7 & y_{10} & y_{11} & x_1 x_2^3 \\
y_5 & y_7 & y_9 & y_{11} & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\ y_4 & y_6 & y_7 & y_{10} & y_{11} & y_{12} \\ y_5 & y_7 & y_9 & y_{11} & y_{12} & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\
y_4 & y_6 & y_7 & y_{10} & y_{11} & y_{12} \\
y_5 & y_7 & y_9 & y_{11} & y_{12} & y_2
\end{pmatrix} \succeq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrl}
A & & -x_1^3 & +x_1 & +2x_2 & -1 & \geq 0 \\
B & -x_2^4 & +2x_1^2 & -2x_1x_2 & +x_2^2 & -\frac{1}{3} & \geq 0 \\
C & & -x_1^2 & -x_2^2 & +x_1 & +4 & \geq 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrl}
A & & -x_1^3 & + x_1 & + 2x_2 & -1 & \geq & 0 \\
B & -x_2^4 & + 2x_1^2 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq & 0 \\
C & & -x_1^2 & - x_2^2 & + x_1 & + 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$(-x_1^2 - x_2^2 + x_1 + 4)(a + bx_1 + cx_2) \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$(-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c \end{pmatrix} (-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \iff$$

$$\begin{pmatrix} a & b & c \end{pmatrix} (-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -x_1^2 - x_2^2 + x_1 + 4 & \ldots & \ldots \\ -x_1^3 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\ -x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \ldots & \ldots \\
-x_1^3 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \ldots & \ldots \\
-y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \ldots & \ldots \\
-y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \ldots & \ldots \\
-y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \ldots & \ldots \\
-y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - x_2^2 + x_1 + 4 & \ldots & \ldots \\
-y_1 - x_1x_2^2 + y_3 + 4x_1 & \ldots & \ldots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - x_2^2 + x_1 + 4 & \ldots & \ldots \\ -y_1 - x_1x_2^2 + y_3 + 4x_1 & \ldots & \ldots \\ -x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc} A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\ B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\ C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - x_2^2 + x_1 + 4 & \ldots & \ldots \\ -y_1 - x_1 x_2^2 + y_3 + 4x_1 & \ldots & \ldots \\ -x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
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C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - x_2^2 + x_1 + 4 & \ldots & \ldots \\ -y_1 - x_1 x_2^2 + y_3 + 4x_1 & \ldots & \ldots \\ -x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

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System of polynomial inequalities

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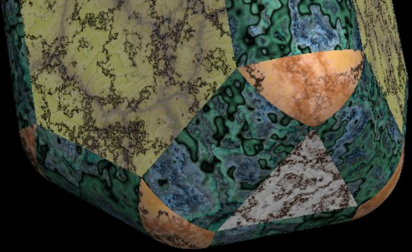

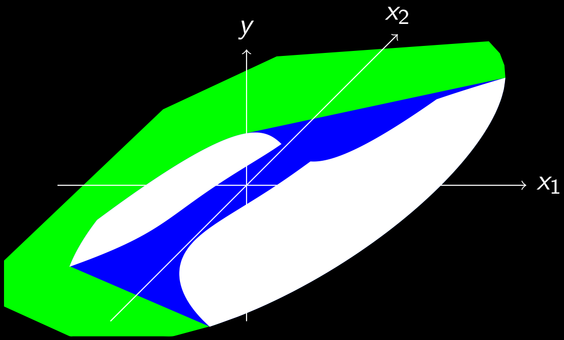

conv S

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  Schmüdgen relaxation

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We have $S \subseteq \operatorname{conv} S \subseteq S' \subseteq \dots \subseteq S'_4 \subseteq S'_3 \subseteq S'_2 \subseteq S'_1$.
The question is whether $\operatorname{conv} S = S'_k$ for some $k \in \mathbb{N}$.

Suppose $S \neq \emptyset$ and fix $k \in \mathbb{N} := \{1, 2, 3, \dots\}$.

Proposition (Powers & Scheiderer 2005).
If S has non-empty interior, then T_k is closed in $\mathbb{R}[\bar{X}]_k$.

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Proposition. If S is compact, then $\operatorname{conv} S$ is closed in $\mathbb{R}^n$.

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Proposition. If $\operatorname{conv} S$ is closed, then
$\operatorname{conv} S = S'_k \iff \forall f \in \mathbb{R}[\bar{X}]_1 : (f \geq 0 \text{ on } S \implies f \in \overline{T_k})$.

Suppose S is compact.

Theorem (Schmüdgen 1991).

(a) $\forall L \in \mathcal{L}$: $\exists$ probability measure μ on S: $\forall p \in \mathbb{R}[\bar{X}]$: $L(p) = \int p \, d\mu$

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Corollary. $\exists c \in \mathbb{N} : \forall k \in \mathbb{N}_{\geq c} : \forall x \in S'_k : \operatorname{dist}(x, \operatorname{conv} S) \leq \frac{c}{\sqrt[c]{k}}$

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Problem: We do not get degree bounds like for Schmüdgen in this way.

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Prestel: Bounds for representations of polynomials positive on compact semi-algebraic sets Valuation theory and its applications, Vol. I (Saskatoon, SK, 1999), 253–260, Fields Inst. Commun., 32, Amer. Math. Soc., 2002 http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.27.3189&rep=rep1&type=pdf

S.: An algorithmic approach to Schmüdgen's Positivstellensatz
J. Pure Appl. Algebra 166 (2002), 307—319
http://perso.univ-rennes1.fr/markus.schweighofer/publications/schmuedgen.pdf
http://dx.doi.org/10.1016/S0022-4049(01)00041-X

S.: On the complexity of Schmüdgen's Positivstellensatz
J. Complexity 20, no. 4 (2004), 529–543
http://arxiv.org/abs/0812.2657
http://dx.doi.org/10.1016/j.jco.2004.01.005

S.: Optimization of polynomials on compact semialgebraic sets
SIAM J. Opt. 15, no. 3 (2005), 805–825
http://perso.univ-rennes1.fr/markus.schweighofer/publications/convergence.pdf
http://dx.doi.org/10.1137/s1052623403431779

Hol & Scherer: Matrix sum-of-squares relaxations for robust semi-definite programs
Math. Prog. 107, no. 1-2 (2006), 189–211
http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.105.7367&rep=rep1&type=pdf
http://dx.doi.org/10.1007/s10107-005-0684-2

Nie & S.: On the complexity of Putinar's Positivstellensatz
J. Complexity 23, no. 1 (2007), 135—150
http://arxiv.org/abs/0812.2657
http://dx.doi.org/10.1007/10.1016/j.jco.2006.07.002

Klep & S.: Pure states, positive matrix polynomials and sums of hermitian squares http://arxiv.org/abs/0907.2260

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Definition. Let $p \in \mathbb{R}[\bar{X}]$ and $U \subseteq \mathbb{R}^n$.

$$p \text{ strictly concave on } U \;:\Longleftrightarrow\; D^2 p \prec 0 \text{ on } U \;\Longleftrightarrow\; \forall x \in U\colon \forall v \in \mathbb{R}^n \setminus \{0\}\colon D^2 p(x)[v,v] < 0$$

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$$p \text{ strictly quasiconcave on } U \;:\Longleftrightarrow\; \forall x \in U \colon \forall v \in \mathbb{R}^n \setminus \{0\} \colon (Dp(x)[v] = 0 \implies D^2 p(x)[v,v] < 0)$$

Suppose S is compact, convex and has non-empty interior.

Lemma (Helton & Nie). If each g_i is strictly concave on S, then $S = S'_k$ for some $k \in \mathbb{N}$.

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Idea of proof. Let $u \in \partial S$ and $f \in \mathbb{R}[\bar{X}]_1 \setminus \{0\}$ with $f \geq 0$ on S and $f(u) = 0$.

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Theorem (Helton & Nie). Suppose each g_i is strictly quasiconcave on $S \cap \{g_i = 0\}$ and a very ugly additional hypothesis is fulfilled that might follow from this. Then $S = S'_k$ for some $k \in \mathbb{N}$.

In the introduction, we have proved the following lemma.

Lemma (Helton & Nie). If $U_1, \ldots, U_\ell \subseteq \mathbb{R}^n$ are bounded semidefinitely representable sets, then so is $\operatorname{conv} \bigcup_{i=1}^{\ell} U_i$.

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One should try to turn this into a symbolic algorithm.

Suppose S is convex and $S^\circ \neq \emptyset$.

Theorem (Netzer & Plaumann & S.) If $S = S'_k$ for some $k \in \mathbb{N}$, then all faces of S are exposed.

Netzer & Plaumann & S.: Exposed faces of semidefinite representable sets http://arxiv.org/abs/0902.3345

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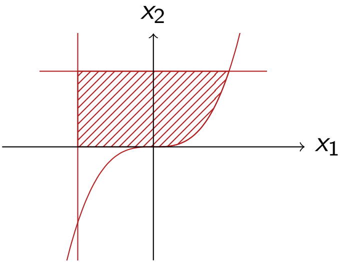

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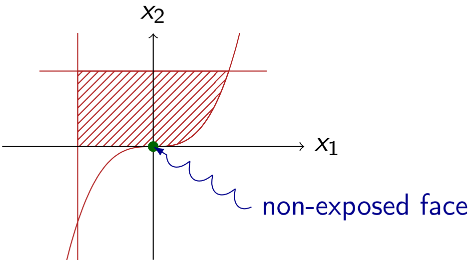

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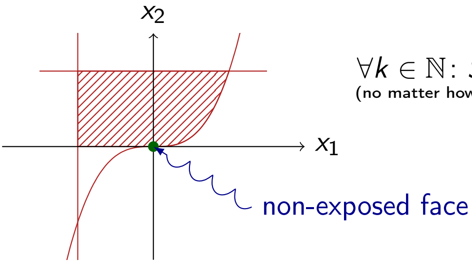

$\forall k \in \mathbb{N}\colon S \neq S'_k$
(no matter how $g_1, \ldots, g_m$ are chosen)

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