Inclusion of spectrahedra, free spectrahedra and coin tossing

(joint work with Bill Helton, Igor Klep and Scott McCullough)

Markus Schweighofer

Universität Konstanz

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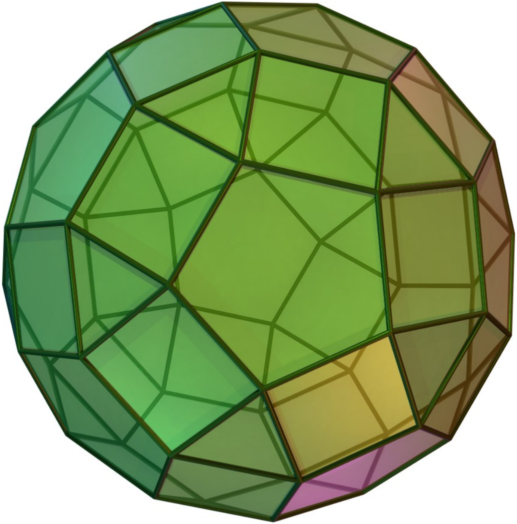
A (closed convex) polyhedron



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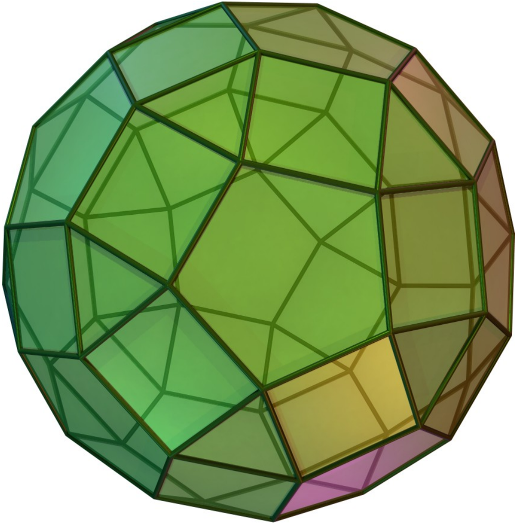
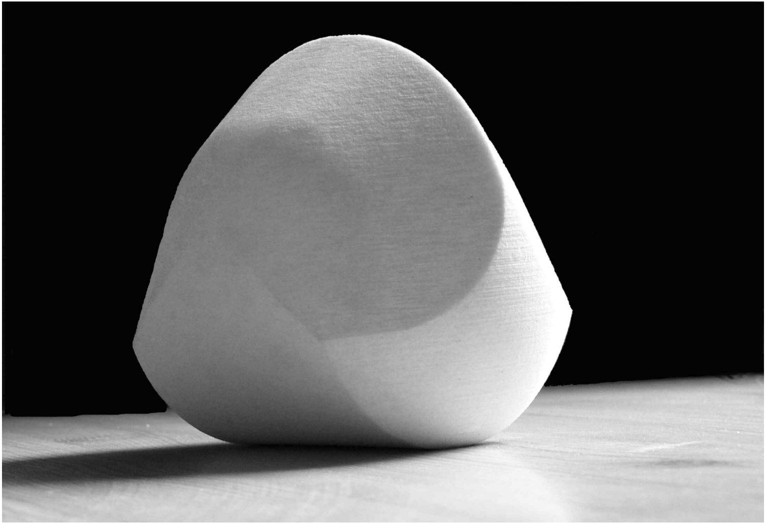


...called rhombicosidodecahedron.

A spectrahedron



Spectrahedra

A pencil (of size d in n variables) is a monic linear symmetric real matrix polynomial

$$\begin{aligned} A &= I_d + A_1\mathtt{x}_1 + \dots A_n\mathtt{x}_n \\ &= \begin{pmatrix} 1 + a_{11}^{(1)}\mathtt{x}_1 + \cdots + a_{11}^{(n)}\mathtt{x}_n & a_{12}^{(1)}\mathtt{x}_1 + \cdots + a_{12}^{(n)}\mathtt{x}_n & \dots \\ a_{21}^{(1)}\mathtt{x}_1 + \cdots + a_{21}^{(n)}\mathtt{x}_n & 1 + a_{22}^{(1)}\mathtt{x}_1 + \cdots + a_{22}^{(n)}\mathtt{x}_n & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \\ &\in \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_n]^{d\times d} = \mathbb{R}[\mathtt{x}]^{d\times d} \end{aligned}$$

where $A_i = (a_{k\ell}^{(i)})_{1\leq k,\ell\leq d} \in S\mathbb{R}^{d\times d}$.

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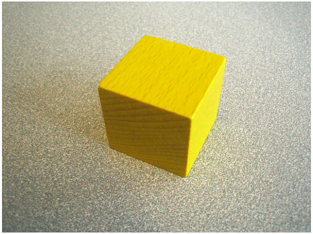
$S_A(1) := \{x \in \mathbb{R}^n \mid A(x) \succeq 0\}$ is the spectrahedron defined by A.

The $S_A(1)$ with A a diagonal pencil are exactly the polyhedra with 0 in their interior.

The cube

$$C_n := \begin{pmatrix} 1+\mathtt{x}_1 & & & & & & \\ & 1-\mathtt{x}_1 & & & & & \\ & & 1+\mathtt{x}_2 & & & & \\ & & & 1-\mathtt{x}_2 & & & \\ & & & & \ddots & & \\ & & & & & 1+\mathtt{x}_n & \\ & & & & & & 1-\mathtt{x}_n \end{pmatrix}$$

defines the cube $S_{C_n}(1) = [-1,1]^n$.



The disk

$$A := \begin{pmatrix} 1 + x_1 & x_2 \\ x_2 & 1 - x_1 \end{pmatrix} \qquad \text{and} \qquad B := \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix}$$

define both the disk

$$S_A(1) = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\} = S_B(1)$$

since $\det A = 1 - x_1^2 - x_2^2 = \det B$.

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It is about detecting inclusion (containment) of two spectrahedra whose interiors contain both 0 (or another known point).

Mainly, it is about detecting inclusion of a cube in a spectrahedron.

It is not about testing emptiness or low-dimensionality of spectrahedra.

Certifying inclusion of spectrahedra

Observation. Let $A \in \mathbb{R}[\mathrm{x}]^{m\times m}$ and $B \in \mathbb{R}[\mathrm{x}]^{d\times d}$ be pencils. If there exist $P \in \mathbb{R}^{d\times d}$ and $Q_i \in \mathbb{R}^{m\times d}$ such that

$$(*) \qquad B = P^*P + \sum_i Q_i^* A Q_i,$$

then $S_A(1) \subseteq S_B(1)$.

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Example. With $A := \begin{pmatrix} 1+\mathrm{x}_1 & \mathrm{x}_2 \\ \mathrm{x}_2 & 1-\mathrm{x}_1 \end{pmatrix}$ and $B := \begin{pmatrix} 1 & \mathrm{x}_1 & \mathrm{x}_2 \\ \mathrm{x}_1 & 1 & 0 \\ \mathrm{x}_2 & 0 & 1 \end{pmatrix}$ from above, we have

$$2B = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} A \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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then $S_A(1) \subseteq S_B(1)$. We will see that the converse fails in general.

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Free spectrahedra

Consider again a pencil

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For $X \in (S\mathbb{R}^{m\times m})^n$

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\begin{aligned}
A(X) &= I_d \otimes I_m + A_1 \otimes X_1 + \dots A_n \otimes X_n \\
&= \begin{pmatrix} I_m + a_{11}^{(1)} X_1 + \cdots + a_{11}^{(n)} X_n & a_{12}^{(1)} X_1 + \cdots + a_{12}^{(n)} X_n & \cdots \\ a_{21}^{(1)} X_1 + \cdots + a_{21}^{(n)} X_n & I_m + a_{22}^{(1)} X_1 + \cdots + a_{22}^{(n)} X_n & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \\
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$S_A(m) := \{X \in (S\mathbb{R}^{m\times m})^n \mid A(X) \succeq 0\}$

$S_A := \bigcup_{m\in\mathbb{N}} S_A(m)$ is the free spectrahedron defined by A.

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$S_A := \bigcup_{m\in\mathbb{N}} S_A(m)$ is the free spectrahedron defined by A.

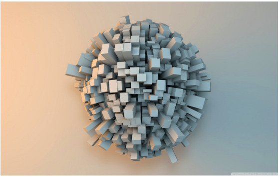
Condition $(*)$ certifies not only $S_A(1) \subseteq S_B(1)$ but even $S_A \subseteq S_B$.

The free cube

$$C_n = \begin{pmatrix} 1+\mathrm{x}_1 & & & & & & \\ & 1-\mathrm{x}_1 & & & & & \\ & & 1+\mathrm{x}_2 & & & & \\ & & & 1-\mathrm{x}_2 & & & \\ & & & & \ddots & & \\ & & & & & 1+\mathrm{x}_n & \\ & & & & & & 1-\mathrm{x}_n \end{pmatrix}$$

defines the free cube

$$\mathscr{C}_n := S_{C_n} = \bigcup_{m \in \mathbb{N}} \left\{ X \in (S\mathbb{R}^{m\times m})^n \mid \|X_i\| \leq 1 \right\}.$$



The free disk

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$$S_B = \bigcup_{m\in\mathbb{N}} \{X \in (S\mathbb{R}^{m\times m})^2 \mid X_1^2 + X_2^2 \preceq I_m\}$$

is the free disk but $S_A \neq S_B$ since

$$\left(\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{3}{4} \\ \frac{3}{4} & 0 \end{pmatrix}\right) \in S_B \setminus S_A.$$

Although we have $S_A(1) = S_B(1)$, we have $S_B \not\subseteq S_A$.

Certifying inclusion of free spectrahedra

Theorem (Helton, Klep, McCullough 2012).
Let $A \in \mathbb{R}[\mathrm{x}]^{m\times m}$ and $B \in \mathbb{R}[\mathrm{x}]^{d\times d}$ be pencils.
Then there exist $P \in \mathbb{R}^{d\times d}$ and $Q_i \in \mathbb{R}^{m\times d}$ such that

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Helton, Klep, McCullough: The matricial relaxation of a linear matrix inequality, Math. Program. 138 (2013), no. 1-2, Ser. A, 401–445 (was first but appeared later)
http://arxiv.org/abs/1003.0908.pdf

Helton, Klep, McCullough: The convex Positivstellensatz in a free algebra, Adv. Math. 231 (2012), no. 1, 516–534
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Kellner, Theobald, Trabandt: Containment problems for polytopes and spectrahedra, SIAM J. Optim. 23 (2013), no. 2, 1000–1020
`http://arxiv.org/abs/1204.4313`

Kellner, Theobald, Trabandt: A Semidefinite Hierarchy for Containment of Spectrahedra
`http://arxiv.org/abs/1308.5076`

Inclusion of free spectrahedra

Theorem. Let $A \in \mathbb{R}[\mathrm{x}]^{m\times m}$ and $B \in \mathbb{R}[\mathrm{x}]^{d\times d}$ be pencils with $S_A = -S_A$ and $S_A(1) \subseteq S_B(1)$. Then $S_A \subseteq dS_B$.

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Example. With $A := \begin{pmatrix} 1 + \mathrm{x}_1 & \mathrm{x}_2 \\ \mathrm{x}_2 & 1 - \mathrm{x}_1 \end{pmatrix}$ and $B := \begin{pmatrix} 1 & \mathrm{x}_1 & \mathrm{x}_2 \\ \mathrm{x}_1 & 1 & 0 \\ \mathrm{x}_2 & 0 & 1 \end{pmatrix}$ from above,

$$S_B \subseteq S_A \subseteq 3S_B.$$

The matrix cube problem

Theorem (Ben Tal, Nemirovski 2002). For $d \in \mathbb{N}$, define $\vartheta(d) \in [1, \infty)$ by

$$\frac{1}{\vartheta(d)} = \min_{\substack{a \in \mathbb{R}^d \\ |a_1| + \cdots + |a_d| = d}} \int_{S^{d-1}} \left| \sum_{i=1}^{d} a_i \xi_i^2 \right| d\xi.$$

Then $\vartheta(1) = 1$, $\vartheta(2) = \frac{\pi}{2}$,
$\vartheta(d) \leq \frac{\pi}{2}\sqrt{d} \leq \sqrt{3d}$ ($\leq \sqrt{d^2} = d$ for $d \geq 3$) and if $A = I + A_1 \mathrm{x}_1 + \cdots + A_n \mathrm{x}_n$ is a pencil with real matrices A_i of rank at most d such that $[-1, 1]^n \subseteq S_A(1)$, then

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Ben-Tal, Nemirovski: On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty, SIAM J. Optim. 12 (2002), no. 3, 811–833

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Our contributions to this theorem:

- The theorem follows naturally from a new dilation theorem.
- Analytic expression for $\vartheta(d)$ for even d and implicit characterization of $\vartheta(d)$ for odd d.
- The scaling factor $\vartheta(d)$ is sharp.

Dilation theorem

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Theorem. Let $d \in \mathbb{N}$. There is a Hilbert space H, an isometry $V: \mathbb{R}^d \to H$ and a set $\mathscr{T}$ of commuting self-adjoint contractions on H such that for each $X \in \mathscr{C}_n(d)$ there exists a $T \in \mathscr{T}$ with $X = \vartheta(d) V^* T V$.

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In the Ben-Tal & Nemirovski theorem, let A be of size d. It was already known that to show $\mathscr{C}_n \subseteq \vartheta(d)S_A$ it suffices to prove $\mathscr{C}_n(d) \subseteq \vartheta(d)S_A(d)$.

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Proof idea: $H := L^2(O(d), \mathbb{R}^n)$, $V\colon \mathbb{R}^d \to H$, $v \mapsto (U \mapsto v)$, $V^*\colon H \to \mathbb{R}^d$, $f \mapsto \int_{O(d)} f(U)\, dU$, $\mathscr{T}$ consists of all operators $T_D\colon H \to H$, $f \mapsto (U \mapsto UD(U)U^* f(U))$ where $D\colon O(d) \to \mathbb{R}^{d \times d}$ is any measurable function taking diagonal contractive values.

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Take $D\colon O(d) \to \mathbb{R}^{d\times d},\ U \mapsto \sum_{i=1}^d \operatorname{sgn}(e_i^*U^*(\lambda + \mu X)Ue_i)e_ie_i^*$ for certain carefully chosen $\lambda, \mu \in \mathbb{R}$. Then $X = \vartheta(d)V^*T_DV$.

Better bounds for $\vartheta(d)$

We considerably improve the upper bound on $\vartheta(d)$ given by Ben Tal and Nemirovski and prove also a lower bound.

Theorem. Let $d \in \mathbb{N}$. If d is even, then

$$\frac{\sqrt{\pi}}{2}\sqrt{d+1} \leq \vartheta(d) \leq \frac{\sqrt{\pi}}{2} \cdot \frac{d}{\sqrt{d-1}}.$$

If $d \neq 1$ is odd, then

$$\sqrt[4]{\left(1-\frac{1}{d+1}\right)^{d+1}\left(1+\frac{1}{d-1}\right)^{d-1}} \cdot \frac{\sqrt{\pi}}{2}\sqrt{d+\frac{3}{2}} \leq \vartheta(d) \leq \frac{\sqrt{\pi}}{2} \cdot \frac{d+2}{\sqrt{d+\frac{5}{2}}}.$$

We have $\displaystyle\lim_{d\to\infty} \frac{\vartheta(d)}{\sqrt{d}} = \frac{\sqrt{\pi}}{2}$.

Computing $\vartheta(d)$

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$$\vartheta_-(d) \leq \vartheta(d) = \frac{\Gamma\left(\frac{d+3}{4}\right)\Gamma\left(\frac{d+5}{4}\right)}{p^{\frac{d-1}{4}}(1-p)^{\frac{d+1}{4}}\Gamma\left(\frac{d}{2}+1\right)} \leq \min\{\vartheta_+(d), \vartheta_{++}(d)\}$$

where $\vartheta_-(d)$, $\vartheta_+(d)$ and $\vartheta_{++}(d)$ are given by

$\vartheta_-(d) = \sqrt[4]{\frac{d^{2d}}{(d+1)^{d+1}(d-1)^{d-1}}}\ \vartheta_{++}(d)$,

$\frac{1}{\vartheta_+(d)} = \frac{d-1}{d} I_{\frac{d+1}{2d}}\left(\frac{d+1}{4}, \frac{d+3}{4}\right) + \frac{d+1}{d} I_{\frac{d-1}{2d}}\left(\frac{d-1}{4}, \frac{d+5}{4}\right) - 1$ and

$\vartheta_{++}(d) = \sqrt{\frac{\pi}{2}}\frac{\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d}{2}+1\right)}$.

d	$\vartheta_{-}(d)$	$\vartheta(d)$	$\vartheta_{+}(d)$	$\vartheta_{++}(d)$
1	–	1	–	–
2	–	1.5708	–	–
3	1.73205	1.73482	1.77064	1.88562
4	–	2	–	–
5	2.15166	2.1527	2.17266	2.26274
6	–	2.35619	–	–
7	2.49496	2.49548	2.50851	2.58599
8	–	2.66667	–	–
9	2.79445	2.79475	2.80409	2.87332
10	–	2.94524	–	–
11	3.064	3.06419	3.07131	3.13453
12	–	3.2	–	–
13	3.31129	3.31142	3.31707	3.37565
14	–	3.43612	–	–
15	3.54114	3.54123	3.54585	3.6007
16	–	3.65714	–	–
17	3.75681	3.75688	3.76076	3.8125
18	–	3.86563	–	–

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Let $d \in \mathbb{N}$ with $d \geq 2$. We have simplified the formula of Ben Tal and Nemirovski

$$\frac{1}{\vartheta(d)} = \min_{\substack{a \in \mathbb{R}^d \\ |a_1| + \cdots + |a_d| = d}} \int_{S^{d-1}} \left| \sum_{i=1}^{d} a_i \xi_i^2 \right| d\xi$$

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and we prove that the inner minimum is assumed at the unique $p_{s,t} \in (0,1)$ satisfying

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Let $s, t \in \mathbb{N}$ such that $s \geq t$ and set $d := s + t$.

Suppose you toss a biased coin d times with probability for heads $\frac{s}{d}$ and probability for tails $\frac{t}{d}$.

What is more likely –
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A paper by Perrin and Redside from 2007 says something even more subtle: The difference grows when $s \notin \{0, d\}$ grows.

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Let $d \in \mathbb{N}$ with $d \geq 2$. Breaking the symmetry in s and t,

$$\frac{1}{\vartheta(d)} = \min_{\substack{s,t\in\mathbb{N} \\ s+t=d \\ s \geq t}} \min_{p\in[0,1]} \left(\frac{2(1-p)sI_{1-p}\left(\frac{t}{2}, 1+\frac{s}{2}\right) + 2ptI_p\left(\frac{s}{2}, 1+\frac{t}{2}\right)}{(1-p)s+pt} - 1 \right)$$

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Let $d \in \mathbb{N}$ with $d \geq 2$. Breaking the symmetry in s and t,

$$\frac{1}{\vartheta(d)} = \min_{\substack{s,t\in\mathbb{N}\\ s+t=d\\ s\geq t}} \min_{p\in[0,1]} \left(\frac{2(1-p)sI_{1-p}\left(\frac{t}{2}, 1+\frac{s}{2}\right) + 2ptI_p\left(\frac{s}{2}, 1+\frac{t}{2}\right)}{(1-p)s+pt} - 1 \right)$$

where the inner minimum is assumed at $p_{s,t} \in (0,1)$ defined by $I_{p_{s,t}}\left(\frac{s}{2}, 1+\frac{t}{2}\right) = I_{1-p_{s,t}}\left(\frac{t}{2}, 1+\frac{s}{2}\right)$. The outer minimum is assumed at $(s,t) = \left(\frac{d}{2}, \frac{d}{2}\right)$ for even d and at $(s,t) = \left(\frac{d+1}{2}, \frac{d-1}{2}\right)$ for odd d but this seems extremely hard to prove.

For example, one ingredient in the proof is that $p_{s,t} \leq \frac{s}{d}$ (assuming $s, t \in \mathbb{N}$, $s+t=d$ and $s \geq t$) which is equivalent to

$$I_{\frac{s}{d}}\left(\frac{s}{2}, 1+\frac{t}{2}\right) \geq I_{\frac{t}{d}}\left(\frac{t}{2}, 1+\frac{s}{2}\right).$$

Simmons' theorem for half integers

Let $s, t \in \mathbb{N}$ such that $s \geq t$ and set $d := s + t$.

It turns out that for even s and t, the inequality

$$I_{\frac{s}{d}}\left(\frac{s}{2}, 1 + \frac{t}{2}\right) \geq I_{\frac{t}{d}}\left(\frac{t}{2}, 1 + \frac{s}{2}\right)$$

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The only proof of Simmons' theorem that somewhat showed potential for generalization to half integers was the one of Perrin and Redside. With a lot of effort we could adapt their idea to find a proof for the half integer case.

Simmons' theorem for reals

Conjecture. For all $s, t \in \mathbb{R}$ such that $s \geq t > 0$, setting $d := s + t$, we have

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With a completely different method, we show the following weakening of Simmons for reals:

Theorem. For all $s, t \in \mathbb{R}$ such that $s \geq t \geq 1$ and $s + t \geq 3$, setting $d := s + t$, we have

$$2I_{\frac{s}{d}}(s,t) + 2(s-t)\frac{s^{s-1}t^{t-1}}{d^d B(s,t)} \geq 1.$$

The median of the Beta distribution

Reminder. For $s, t \in \mathbb{R}_{>0}$, the beta distribution $\mathrm{Beta}(s, t)$ is the probability distribution on $[0, 1]$ with density $x \mapsto \frac{x^{s-1}x^{t-1}}{B(s,t)}$ and cumulative density $x \mapsto I_x(s, t)$.

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From the weakening of Simmons' for reals, we deduce:

Theorem. For $s, t \in \mathbb{R}$ with $s \geq t \geq 1$ and $s + t \geq 3$, setting $d := s + t$, the median of $\mathrm{Beta}(s,t)$ lies between $\frac{s}{d}$ and $\frac{s}{d} + \frac{s-t}{d^2}$.

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s	t	$\frac{s}{d}$	median	$\frac{s}{d} + \frac{s-t}{2d^2}$	$\frac{s}{d} + \frac{s-t}{d^2}$	$\frac{s-1}{s-t-2}$
2.5	1	0.714286	0.757858	0.77551	0.836735	1
3	1	0.75	0.793701	0.8125	0.875	1
3	2	0.6	0.614272	0.62	0.64	0.666667
4	2	0.666667	0.68619	0.694444	0.722222	0.75

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If you obtain at least s times head, you pay me t dollars.
If you obtain at least t times tail, you pay me s dollars.
(Consequently, if you obtain exactly s times head, then you pay d dollars in total.)

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Which coin should you choose to minimize the expected loss?