New results on the exactness of Lasserre relaxations for compact basic closed semialgebraic sets

(joint work with Tom-Lukas Kriel)

Semidefinite and Matrix Methods
for Optimization and Communications
Workshop on Positive Semidefinite Rank
National University of Singapore

February 1–5, 2016

Markus Schweighofer

Universität Konstanz

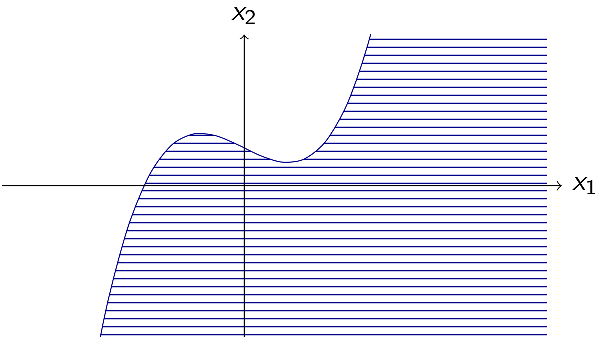
System of polynomial inequalities

$$\begin{array}{rrrrrrrrrrrrr} & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\ - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\ & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \end{array}$$

System of polynomial inequalities

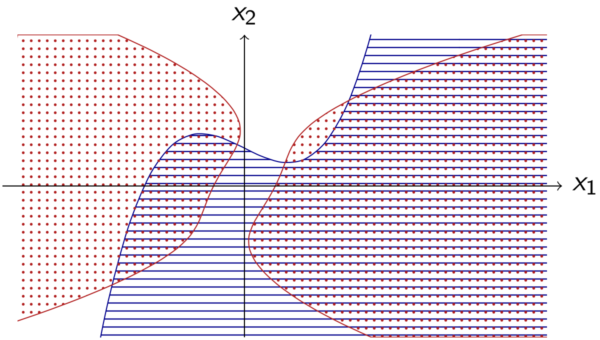
A

$$\begin{array}{rrrrrrrrrrrr} & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\ - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\ & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \end{array}$$



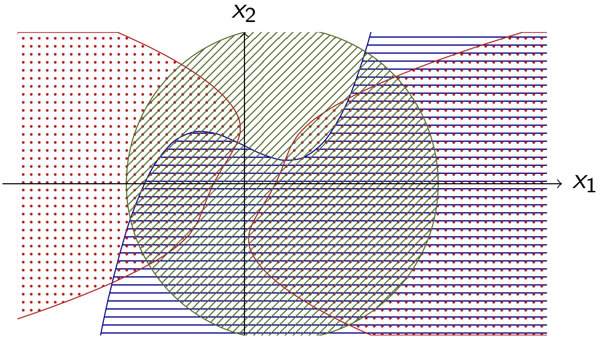
System of polynomial inequalities

$$
\begin{array}{lrcl}
A & x_1^3 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
 & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$



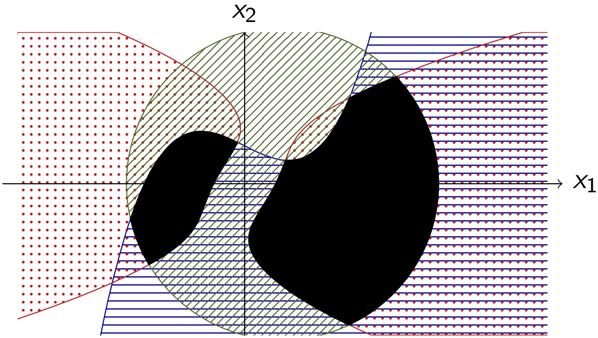
System of polynomial inequalities

$$\begin{array}{lrcl} A & x_1^3 - x_1 - 2x_2 + 1 & \geq & 0 \\ B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\ C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0 \end{array}$$



System of polynomial inequalities

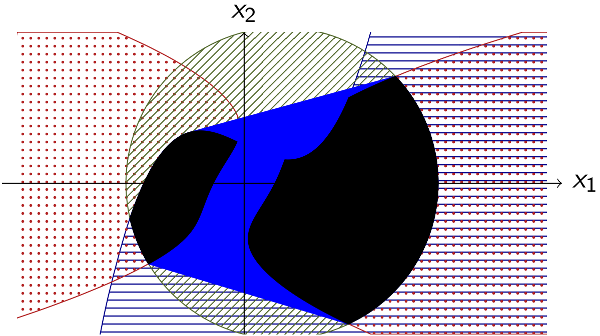
$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



S

System of polynomial inequalities

$$
\begin{array}{lrrrrrrrr}
A & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 \geq 0 \\
B & -x_2^4 & + \; 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} \geq 0 \\
C & & -x_1^2 & - & x_2^2 & + & x_1 & + & 4 \geq 0
\end{array}
$$



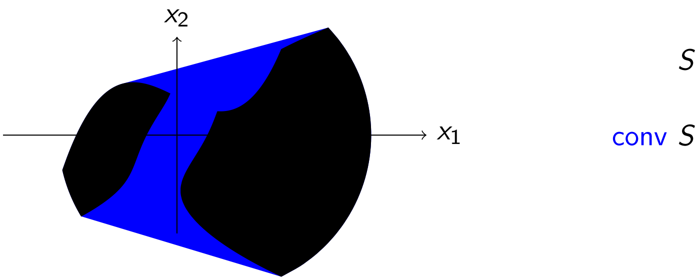
S

conv S

System of polynomial inequalities

Very naive linearization

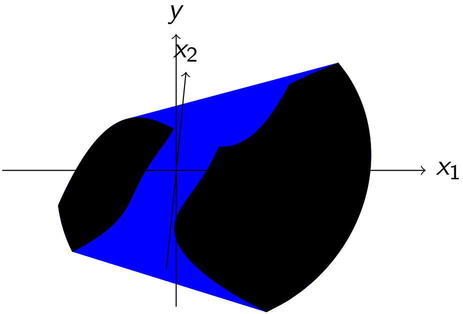
$$
\begin{array}{ll}
A & x_1^3 - x_1 - 2x_2 + 1 \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$



System of polynomial inequalities

Very naive linearization

$$
\begin{array}{llr}
A & & x_1^3 - x_1 - 2x_2 + 1 \geq 0 \\
B & - x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0 \\
C & - x_1^2 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$



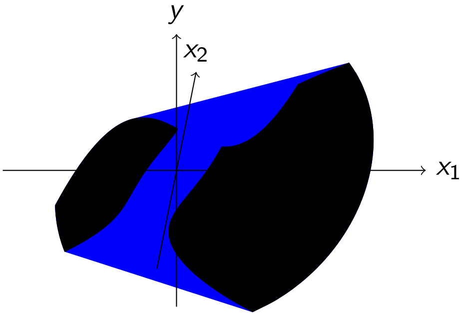
S

conv S

System of polynomial inequalities

Very naive linearization

A	$x_1^3 - x_1 - 2x_2 + 1$	≥ 0
B	$-x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3}$	≥ 0
C	$-x_1^2 - x_2^2 + x_1 + 4$	≥ 0



S

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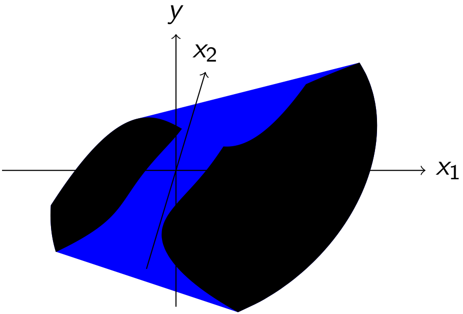
System of polynomial inequalities

Very naive linearization

A	$x_1^3 - x_1 - 2x_2 + 1 \geq 0$
B	$-x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0$
C	$-x_1^2 - x_2^2 + x_1 + 4 \geq 0$



S

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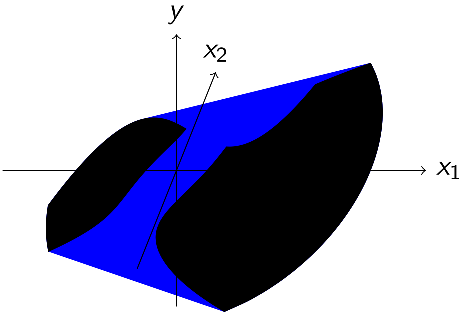
System of polynomial inequalities

Very naive linearization

A	$x_1^3 - x_1 - 2x_2 + 1 \geq 0$
B	$-x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0$
C	$-x_1^2 - x_2^2 + x_1 + 4 \geq 0$



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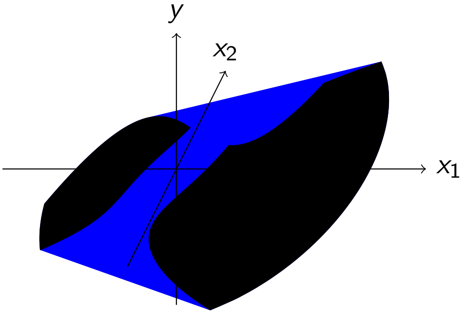
System of polynomial inequalities

Very naive linearization

A	$x_1^3 - x_1 - 2x_2 + 1 \geq 0$
B	$-x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0$
C	$-x_1^2 - x_2^2 + x_1 + 4 \geq 0$



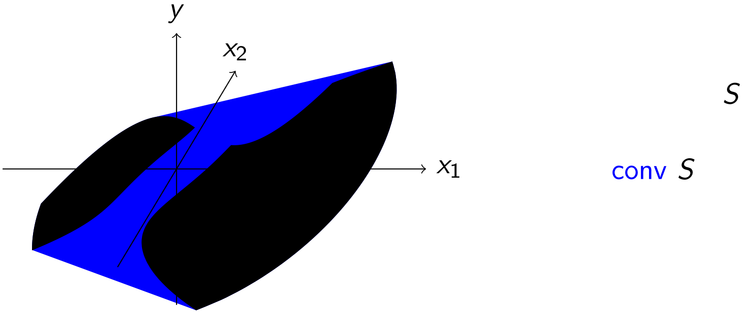
S

conv S

System of polynomial inequalities

Very naive linearization

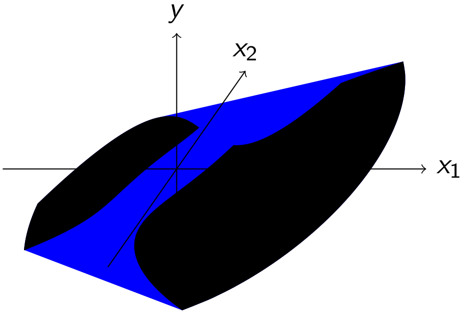
$$
\begin{array}{lrl}
A & x_1^3 - x_1 - 2x_2 + 1 & \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq 0
\end{array}
$$



System of polynomial inequalities

Very naive linearization

$$
\begin{array}{llcrcrcrcrcl}
A & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



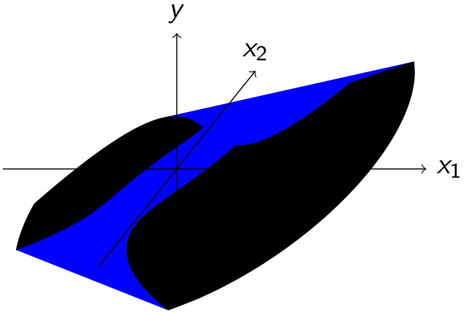
S

conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



S

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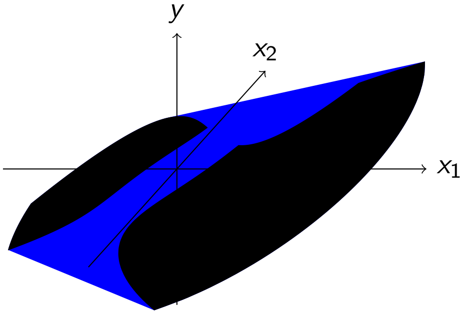
System of polynomial inequalities

Very naive linearization

$$
\begin{array}{llr}
A & & x_1^3 - x_1 - 2x_2 + 1 \geq 0 \\
B & & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0 \\
C & & -x_1^2 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$

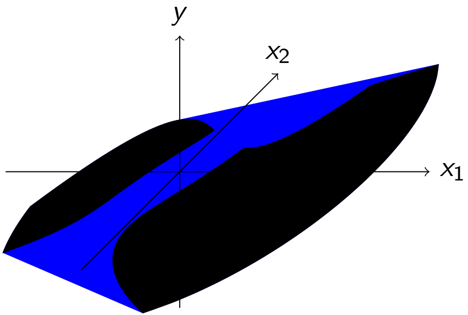


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lcccccccccc}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

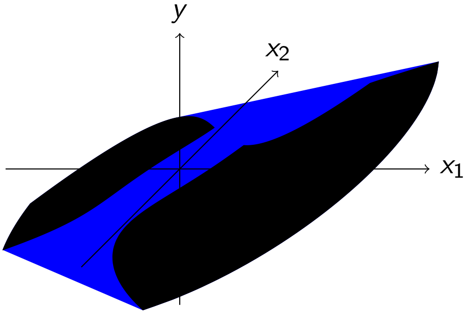


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

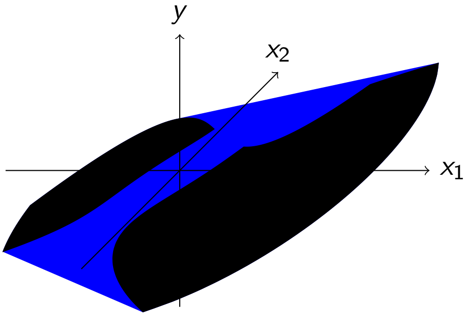


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrcl}
A & y_1 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$

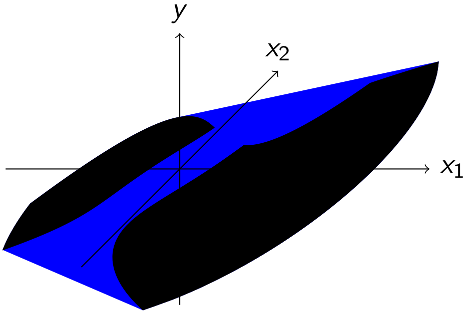


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrrrrrrr}
A & & y_1 & - x_1 & - 2x_2 & + 1 & \geq & 0 \\
B & - y_2 & + 2x_1^2 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq & 0 \\
C & & - x_1^2 & - x_2^2 & + x_1 & + 4 & \geq & 0
\end{array}
$$

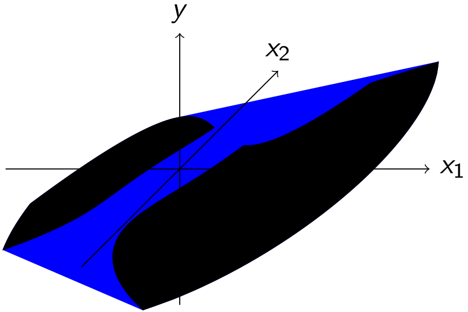


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrrrrrrl}
A & & y_1 - x_1 - 2x_2 + 1 & \geq 0 \\
B & -y_2 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & & \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & & \geq 0
\end{array}
$$

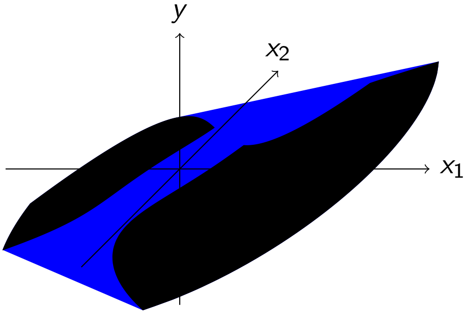


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrrrrrrl}
A & & & y_1 & - \; x_1 & - \; 2x_2 & + \; 1 & \geq 0 \\
B & - \; y_2 & + \; 2y_3 & & - \; 2x_1x_2 & + \; x_2^2 & - \; \frac{1}{3} & \geq 0 \\
C & & - \; y_3 & & - \; x_2^2 & + \; x_1 & + \; 4 & \geq 0
\end{array}
$$

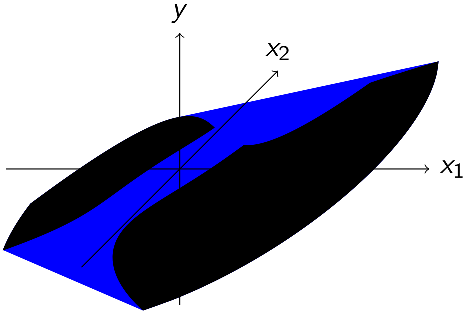


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrrrrrrrr}
A & & & y_1 & - x_1 & - 2x_2 & + 1 & \geq & 0 \\
B & -y_2 & + & 2y_3 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq & 0 \\
C & & - & y_3 & - x_2^2 & + x_1 & + 4 & \geq & 0
\end{array}
$$

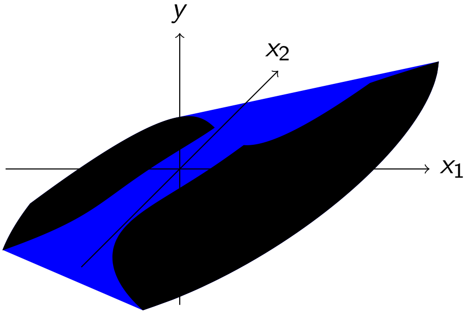


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lr}
A & y_1 - x_1 - 2x_2 + 1 \geq 0 \\
B & -y_2 + 2y_3 - 2y_4 + x_2^2 - \frac{1}{3} \geq 0 \\
C & -y_3 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$

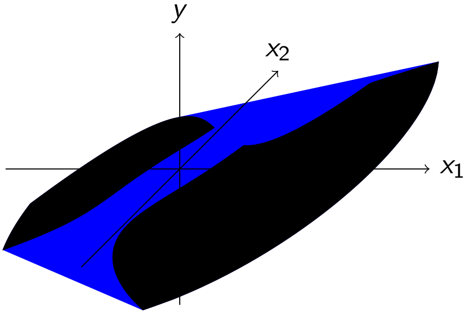


conv S

System of polynomial inequalities

Very naive linearization

$$
\begin{array}{lrrrrrrl}
A & & & y_1 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & - y_2 + 2y_3 - 2y_4 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & - y_3 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$

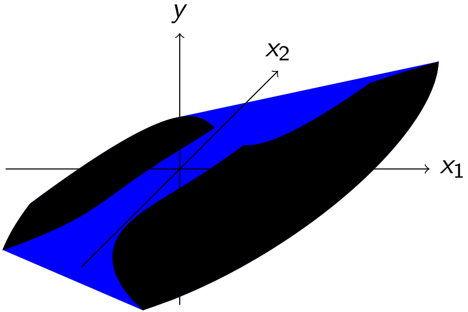


conv S

System of linear inequalities

Very naive linearization

$$
\begin{array}{lrrrrrrrrrl}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 \geq 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} \geq 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 \geq 0
\end{array}
$$

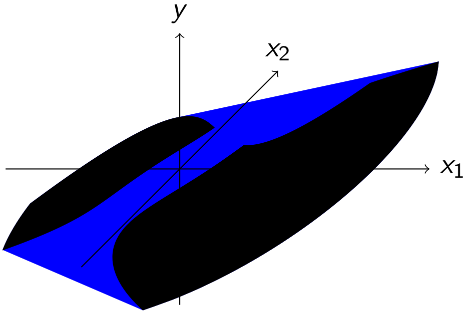


conv S

System of linear inequalities

Very naive linearization

$$\begin{array}{llcl}
A & y_1 - x_1 - 2x_2 + 1 & \geq & 0 \\
B & -y_2 + 2y_3 - 2y_4 + y_5 - \frac{1}{3} & \geq & 0 \\
C & -y_3 - y_5 + x_1 + 4 & \geq & 0
\end{array}$$



conv S

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrr}
A & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 \geq 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} \geq 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 \geq 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant:

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} & & & & & & & & & & & & \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

A				x_1^3	$-$	x_1	$-$	$2x_2$	$+$	1	$\geq$	0
B	$-$	x_2^4	$+$	$2x_1^2$	$-$	$2x_1x_2$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	x_1^2	$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0
redundant:												
AB	$-$	$x_1^3x_2^4$	$+$	$\ldots$	$+$	x_2^2	$+$	$\frac{2}{3}x_2$	$-$	$\frac{1}{3}$	$\geq$	0
AC		x_1^5	$+$	$\ldots$	$-$	x_1	$+$	$8x_2$	$-$	4	$\geq$	0
ABC	$-$	$x_1^5x_2^4$	$+$	$\ldots$	$-$	$\frac{13}{3}x_2^2$	$-$	$\frac{8}{3}x_2$	$+$	$\frac{4}{3}$	$\geq$	0

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} & & & & & & & & & & & & \\
AB & - & x_1^3x_2^4 & + & \ldots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

A					y_1	$-$	x_1	$-$	$2x_2$	$+$	1	$\geq$	0
B	$-$	x_2^4	$+$	$2x_1^2$	$-$	$2x_1x_2$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	x_1^2	$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0
irredundant:												
AB	$-$	$x_1^3x_2^4$	$+$	$\ldots$	$+$	x_2^2	$+$	$\frac{2}{3}x_2$	$-$	$\frac{1}{3}$	$\geq$	0
AC		x_1^5	$+$	$\ldots$	$-$	x_1	$+$	$8x_2$	$-$	4	$\geq$	0
ABC	$-$	$x_1^5x_2^4$	$+$	$\ldots$	$-$	$\frac{13}{3}x_2^2$	$-$	$\frac{8}{3}x_2$	$+$	$\frac{4}{3}$	$\geq$	0
D^2						x_1^2	$-$	$2x_1x_2$	$+$	x_2^2	$\geq$	0
D^2C	$-$	x_1^4	$+$	$\ldots$	$+$	$4x_1^2$	$+$	$4x_1x_2$	$+$	$4x_2^2$	$\geq$	0

System of polynomial inequalities

Less naive linearization

A					y_1	$-$	x_1	$-$	$2x_2$	$+$	1	$\geq$	0
B	$-$	x_2^4	$+$	$2x_1^2$		$-$	$2x_1x_2$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	x_1^2		$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0
irredundant:													
AB	$-$	$x_1^3x_2^4$	$+$	$\ldots$		$+$	x_2^2	$+$	$\frac{2}{3}x_2$	$-$	$\frac{1}{3}$	$\geq$	0
AC		x_1^5	$+$	$\ldots$		$-$	x_1	$+$	$8x_2$	$-$	4	$\geq$	0
ABC	$-$	$x_1^5x_2^4$	$+$	$\ldots$		$-$	$\frac{13}{3}x_2^2$	$-$	$\frac{8}{3}x_2$	$+$	$\frac{4}{3}$	$\geq$	0
D^2							x_1^2	$-$	$2x_1x_2$	$+$	x_2^2	$\geq$	0
D^2C	$-$	x_1^4	$+$	$\ldots$		$+$	$4x_1^2$	$+$	$4x_1x_2$	$+$	$4x_2^2$	$\geq$	0

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & - & x_1^3x_2^4 & + & \ldots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \ldots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lcrcrcrcrcrcc}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lcrcrcrcrcrcc}
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & & \geq & 0 \\
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} & & & & & & & & & & & & \\
AB & - & x_1^3x_2^4 & + & \dots & + & x_2^2 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & x_1^3 x_2^4 & + & \ldots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \ldots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & x_1^3x_2^4 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrl}
A & & & y_1 & - x_1 & - 2x_2 & + 1 & \geq 0 \\
B & - y_2 & + 2y_3 & - 2y_4 & + y_5 & - \frac{1}{3} & & \geq 0 \\
C & & - y_3 & - y_5 & + x_1 & + 4 & & \geq 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrl}
AB & - y_6 & + \dots & + y_5 & + \frac{2}{3}x_2 & - \frac{1}{3} & \geq 0 \\
AC & x_1^5 & + \dots & - x_1 & + 8x_2 & - 4 & \geq 0 \\
ABC & - x_1^5 x_2^4 & + \dots & - \frac{13}{3}y_5 & - \frac{8}{3}x_2 & + \frac{4}{3} & \geq 0 \\
D^2 & & & y_3 & - 2y_4 & + y_5 & \geq 0 \\
D^2C & - x_1^4 & + \dots & + 4y_3 & + 4y_4 & + 4y_5 & \geq 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrr}
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lcrcrcrcrcrcl}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\multicolumn{13}{l}{\text{irredundant:}} \\
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Less naive linearization

A					y_1	$-$	x_1	$-$	$2x_2$	$+$	1	$\geq$	0
B	$-$	y_2	$+$	$2y_3$	$-$	$2y_4$	$+$	y_5	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	y_3	$-$	y_5	$+$	x_1	$+$	4	$\geq$	0
irredundant:												
AB	$-$	y_6	$+$	$\dots$	$+$	y_5	$+$	$\frac{2}{3}x_2$	$-$	$\frac{1}{3}$	$\geq$	0
AC		y_{10}	$+$	$\dots$	$-$	x_1	$+$	$8x_2$	$-$	4	$\geq$	0
ABC	$-$	y_{13}	$+$	$\dots$	$-$	$\frac{13}{3}y_5$	$-$	$\frac{8}{3}x_2$	$+$	$\frac{4}{3}$	$\geq$	0
D^2						y_3	$-$	$2y_4$	$+$	y_5	$\geq$	0
D^2C	$-$	x_1^4	$+$	$\dots$	$+$	$4y_3$	$+$	$4y_4$	$+$	$4y_5$	$\geq$	0

System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrrl}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrl}
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & y_{13} & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

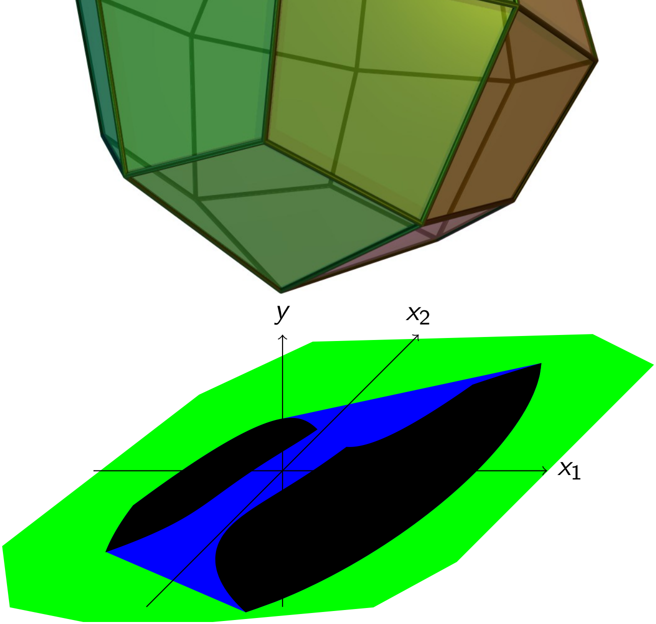
System of polynomial inequalities

Less naive linearization

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & & \geq & 0
\end{array}
$$

irredundant:

$$
\begin{array}{lrrrrrrrrrrrrr}
AB & - & y_6 & + & \dots & + & y_5 & + & \frac{2}{3}x_2 & - & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & y_{13} & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & y_{18} & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$



conv S

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrr}
A & & x_1^3 & - x_1 & - 2x_2 & + 1 & \geq & 0 \\
B & - x_2^4 & + 2x_1^2 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq & 0 \\
C & & - x_1^2 & - x_2^2 & + x_1 & + 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0 \qquad \Longleftrightarrow$$

$$(a+bx_1+cx_2+dx_1^2+ex_1x_2+fx_2^2)\begin{pmatrix}1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\\e\\f\end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lcccccccccccc}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0 \qquad \Longleftrightarrow$$

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lcccccccccccc}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lrrrrrrl}
A & & y_1 & - x_1 & - 2x_2 & + 1 & \geq & 0 \\
B & - y_2 & + 2x_1^2 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq & 0 \\
C & & - x_1^2 & - x_2^2 & + x_1 & + 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lcccccccccccc} A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\ B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\ C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lcccccccccccccc}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & & - & y_2 & + & & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & & - & & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & x_1x_2 & x_2^2 \\ x_1 & y_3 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & x_1x_2 & x_2^2 \\
x_1 & y_3 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & x_2^2 \\
x_1 & y_3 & y_4 & y_1 & x_1^2 x_2 & x_1 x_2^2 \\
x_2 & y_4 & x_2^2 & x_1^2 x_2 & x_1 x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2 x_2 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & x_1^2 x_2 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
x_2^2 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & x_2^2 \\
x_1 & y_3 & y_4 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & y_4 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
y_4 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & x_1^2 x_2 & x_1 x_2^2 \\ x_2 & y_4 & y_5 & x_1^2 x_2 & x_1 x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2 x_2 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & x_1^2 x_2 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & y_4 & y_5 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
y_4 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
y_5 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & x_1x_2^2 \\ x_2 & y_4 & y_5 & y_6 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & y_6 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ y_4 & y_6 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ y_5 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrrl}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & x_1 x_2^2 \\ x_2 & y_4 & y_5 & y_6 & x_1 x_2^2 & x_2^3 \\ y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & y_6 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrl}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\ y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrrrrrr}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\
y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{llllllllllllll}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\ y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\
y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lcccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & x_1^3x_2 & x_1^2x_2^2 \\ y_4 & y_6 & y_7 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ y_5 & y_7 & y_9 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lcccccccccccc} A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\ B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\ C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & y_{10} & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lrrrrrrl}
A & & & y_1 & - x_1 & - 2x_2 & + 1 & \geq 0 \\
B & - y_2 & + 2y_3 & - 2y_4 & + y_5 & - \frac{1}{3} & & \geq 0 \\
C & & - y_3 & - y_5 & + x_1 & + 4 & & \geq 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & y_{10} & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lcccccccccccc}
A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\
y_4 & y_6 & y_7 & y_{10} & y_{11} & x_1 x_2^3 \\
y_5 & y_7 & y_9 & y_{11} & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\
y_4 & y_6 & y_7 & y_{10} & y_{11} & x_1 x_2^3 \\
y_5 & y_7 & y_9 & y_{11} & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{llllllllllll}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\ y_4 & y_6 & y_7 & y_{10} & y_{11} & y_{12} \\ y_5 & y_7 & y_9 & y_{11} & y_{12} & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\
y_4 & y_6 & y_7 & y_{10} & y_{11} & y_{12} \\
y_5 & y_7 & y_9 & y_{11} & y_{12} & y_2
\end{pmatrix} \succeq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$
(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow
$$

$$
(-x_1^2 - x_2^2 + x_1 + 4)(a + bx_1 + cx_2) \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$(-x_1^2 - x_2^2 + x_1 + 4)\begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}\begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lrrrrrrrrrrr}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c \end{pmatrix} (-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lcccccccccccccc}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c \end{pmatrix} (-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \ldots & \ldots \\
-x_1^3 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lcccccccccccc}
A & & & & x_1^3 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \dots & \dots \\
-x_1^3 - x_1x_2^2 + x_1^2 + 4x_1 & \dots & \dots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

Lasserre relaxation

Systematic linearization

$$
\begin{array}{lcccccccccccccc}
A & & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \ldots & \ldots \\
-y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

Lasserre relaxation

Systematic linearization

$$\begin{array}{lcccccccccccc} A & & & & y_1 & - & x_1 & - & 2x_2 & + & 1 & \geq & 0 \\ B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\ C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \end{array}$$

irredundant families (parametrized by $a, b, c, \dots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -x_1^2 - x_2^2 + x_1 + 4 & \dots & \dots \\ -y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \dots & \dots \\ -x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \dots & \dots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

Lasserre relaxation

Systematic linearization

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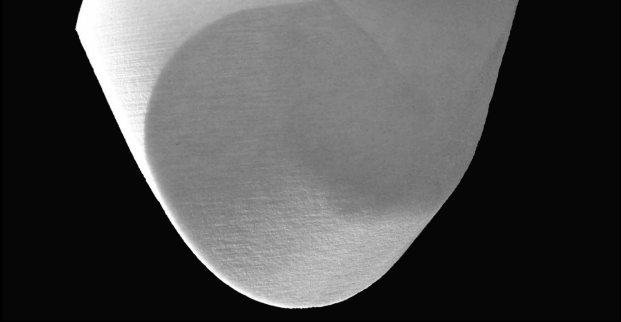
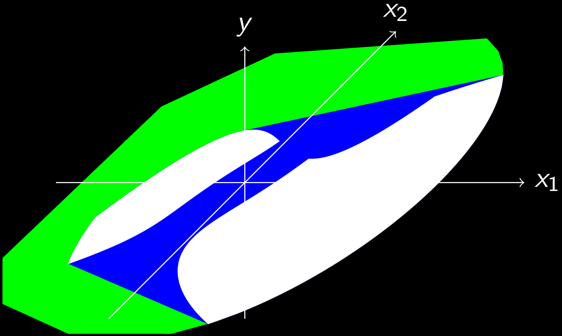
$$





conv S

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  basic closed semialgebraic set

Quadratic modules

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$$M := \left\{ \sum_{i=0}^{m} \sum_{j} p_{ij}^2 g_i \mid p_{ij} \in \mathbb{R}[\underline{X}] \right\} \subseteq \mathbb{R}[\underline{X}].$$

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$$M_d := \left\{ \sum_{i=0}^{m} \sum_{j} p_{ij}^2 g_i \mid p_{ij} \in \mathbb{R}[\underline{X}]_k, 2k + \deg(g_i) \leq d \right\} \subseteq M \cap \mathbb{R}[\underline{X}]_d.$$

Exactness of Lasserre relaxations

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The question whether the Lasserre relaxation is eventually exact, therefore is equivalent to the existence of $d \in \mathbb{N}_0$ such that
$\{\ell \in \mathbb{R}[\underline{X}]_1 \mid \ell \geq 0 \text{ on } S\} \subseteq M_d$

The archimedean condition

Definition. Call M archimedean if $\forall p \in \mathbb{R}[\underline{X}] : \exists N \in \mathbb{N} : p + N \in M$.

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Proof. "$\Longrightarrow$" is trivial, "$\Longleftarrow$" is tricky but easy.

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Proposition. M archimedean $\iff \exists N \in \mathbb{N} : N - \sum_{i=1}^{n} {X_i}^2 \in M$.

It is trivial that M archimedean $\implies$ S compact and one can show the following: The converse is not always but in really a lot of cases true,

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Conversely, when M is archimedean, then S is compact but is the Lasserre relaxation eventually exact?

First main result

S not necessarily connected with not necessarily convex connected components

Definition. We call $p \in \mathbb{R}[\underline{X}]$ strictly quasiconcave at a point $x \in \mathbb{R}^n$ if

$$\forall y \in \mathbb{R}^n : (p'(x)y = 0 \implies y^T p''(x)y < 0).$$

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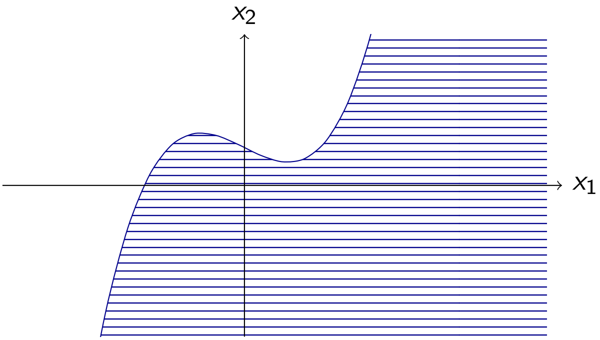
$$p_1 \quad := \quad {X_1}^3 - X_1 - 2X_2 + 1$$



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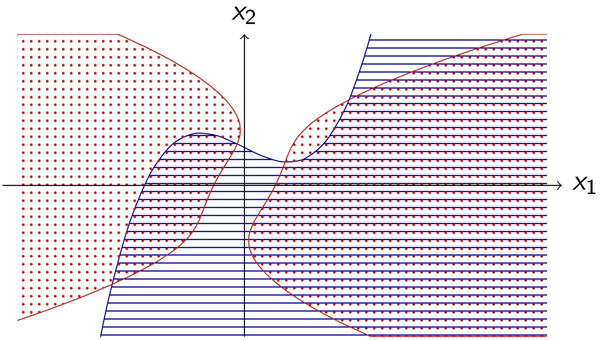

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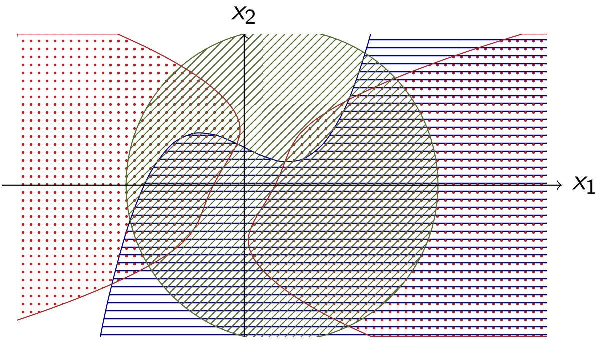

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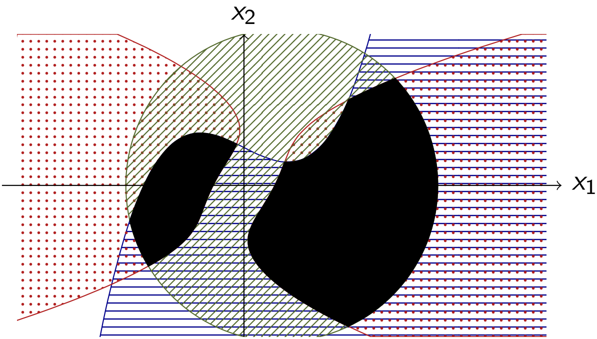

S

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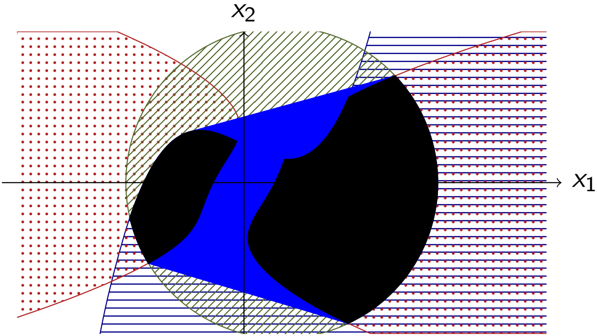

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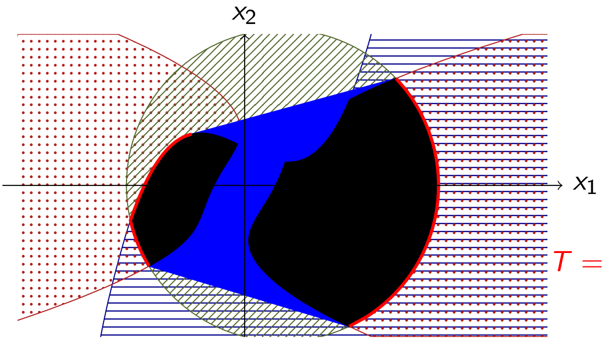

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$T = S \cap \partial \operatorname{conv} S$

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Basic proof strategy. We have to show that there exists $d \in \mathbb{N}_0$ such that $\{\ell \in \mathbb{R}[\underline{X}]_1 \mid \ell \geq 0 \text{ on } S\} \subseteq M_d$.

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For all real closed extension fields R of $\mathbb{R}$, setting $\tilde{S} := \{x \in R^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$ and

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A real closed field is a field admitting a field order such that the intermediate value theorem holds for polynomials. Such an order can be shown to be unique. The prototype of all real closed fields is $\mathbb{R}$. All proper real closed field extensions of $\mathbb{R}$ however admit "infinite" and therefore "infinitesimal" elements we have to cope with.

Second main result

S convex

Definition. We call $p \in \mathbb{R}[\underline{X}]$ g-sos-concave if there exist matrices $P_0, \ldots, P_m$ with entries in $\mathbb{R}[\underline{X}]$ and m columns each such that

$$-p'' = \sum_{i=0}^{n} g_i P_i^T P_i.$$

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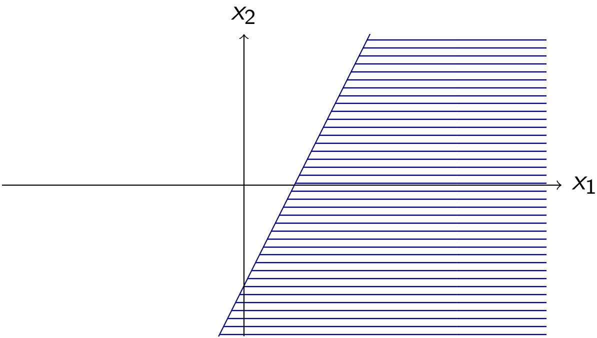
$$p_1 \quad := \quad 6X_1 - 3X_2 - 4$$



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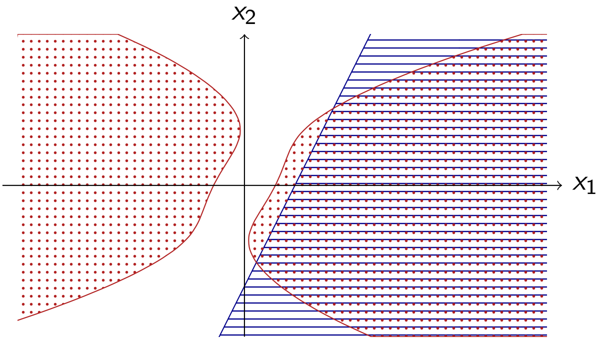

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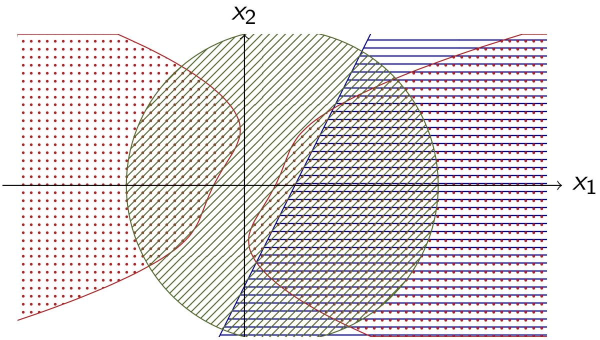

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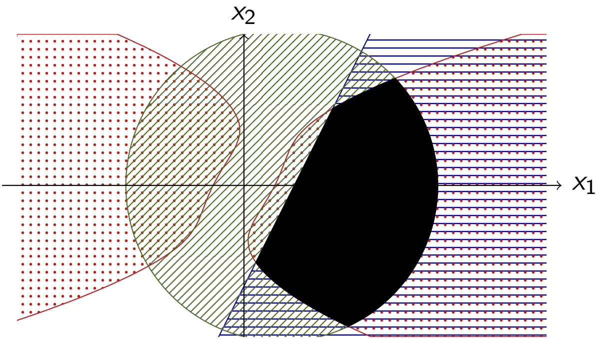

S

An example with no exact Lasserre relaxation

Theorem [Netzer & Plaumann & S. 2010] Suppose S is convex with non-empty interior. If S has at least one non-exposed face, then no Lasserre relaxation of S is exact.

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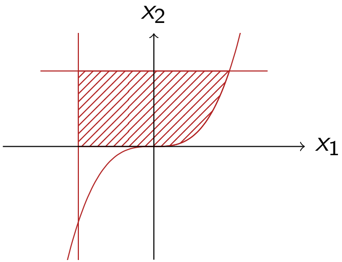

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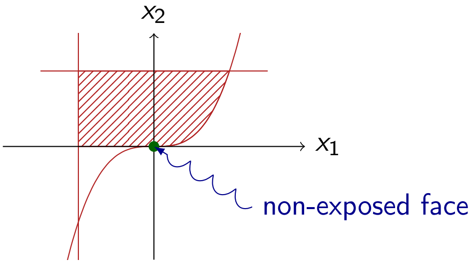

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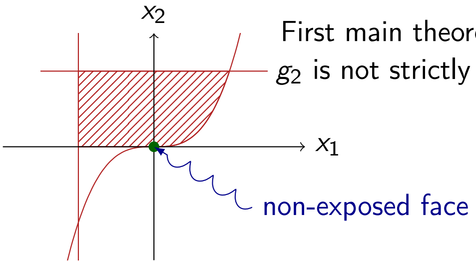

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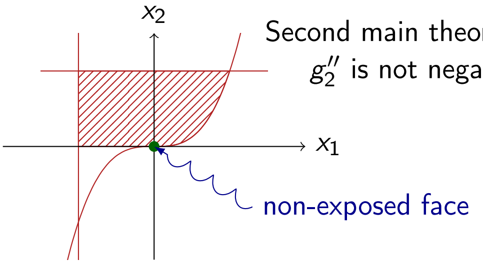

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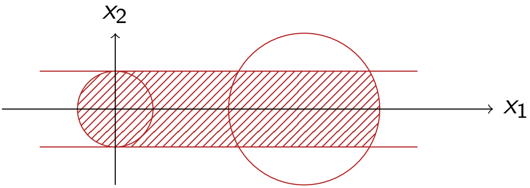

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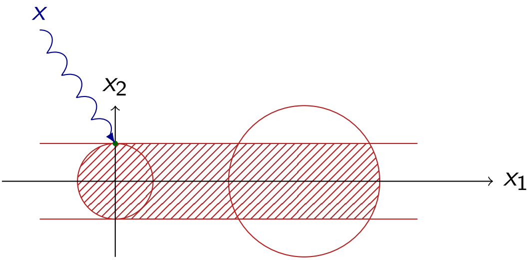

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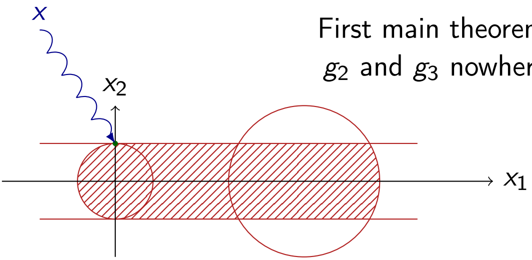

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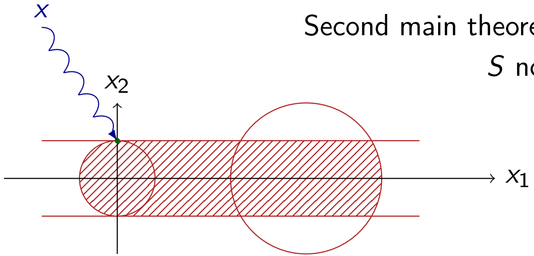

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Outlook

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Theorem [Scheiderer]. True for $n = 2$.

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