Can a system of polynomial inequalities be written as a linear matrix inequality?

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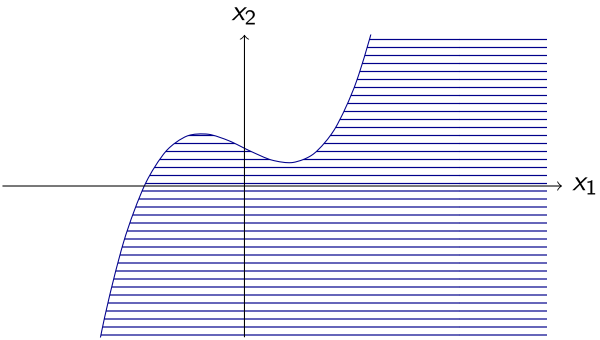
System of polynomial inequalities

$$
\begin{array}{rrrrrrrrrrrrr}
 & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
- & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
 & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

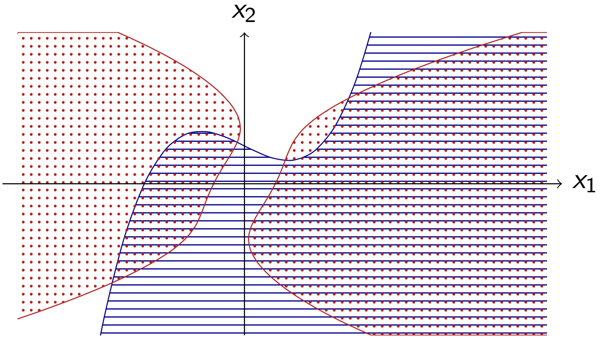
System of polynomial inequalities

$$A \qquad \begin{array}{rrrrrrl} & - x_1^3 & + x_1 & + 2x_2 & - 1 & \geq & 0 \\ - x_2^4 & + 2x_1^2 & - 2x_1x_2 & + x_2^2 & - \frac{1}{3} & \geq & 0 \\ & - x_1^2 & - x_2^2 & + x_1 & + 4 & \geq & 0 \end{array}$$



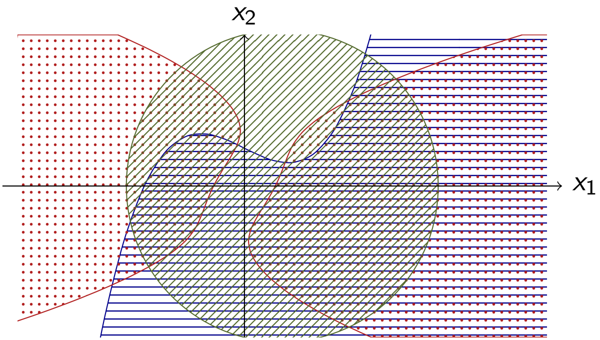
System of polynomial inequalities

$$
\begin{array}{ll}
A & -x_1^3 + x_1 + 2x_2 - 1 \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0 \\
 & -x_1^2 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$



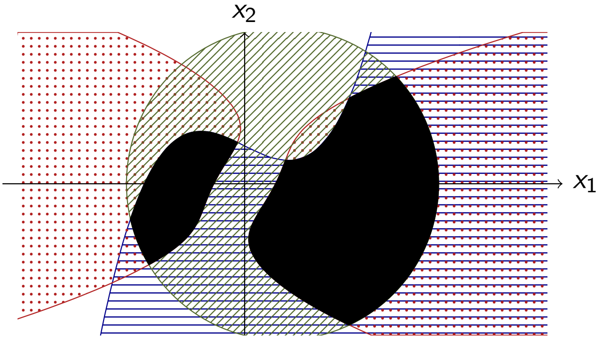
System of polynomial inequalities

$$
\begin{array}{llcccccccc}
A & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 \geq 0 \\
B & - x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} \geq 0 \\
C & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 \geq 0
\end{array}
$$



System of polynomial inequalities

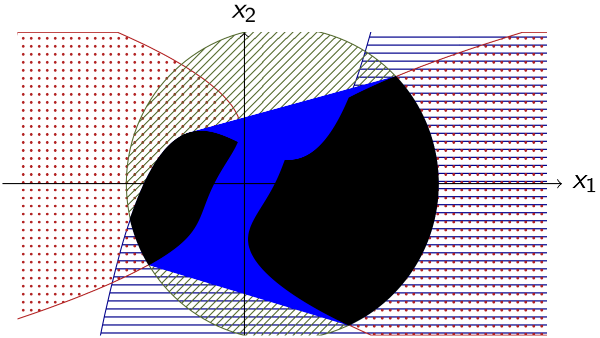
$$
\begin{array}{lr}
A & -x_1^3 + x_1 + 2x_2 - 1 \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \tfrac{1}{3} \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$



S

System of polynomial inequalities

$$\begin{array}{lrl}
A & -x_1^3 + x_1 + 2x_2 - 1 & \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq 0
\end{array}$$

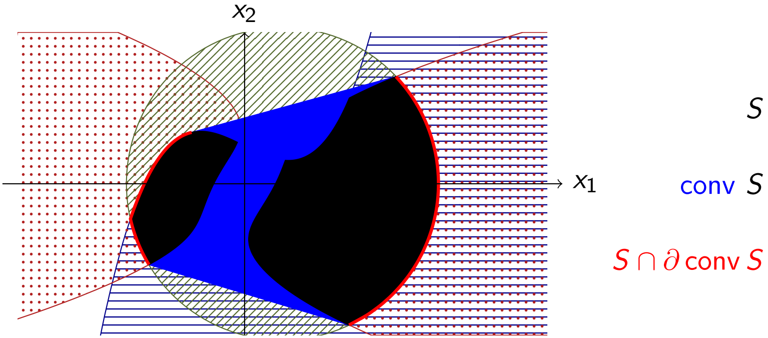


S

conv S

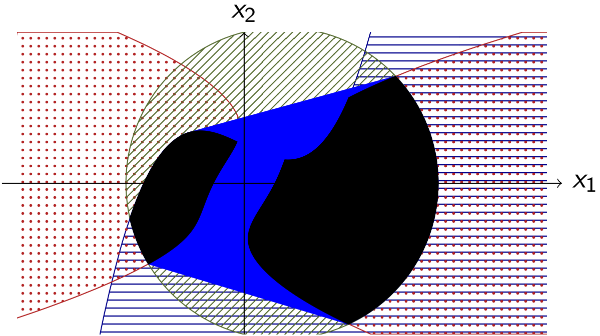
System of polynomial inequalities

$$
\begin{array}{lrcl}
A & -x_1^3 + x_1 + 2x_2 - 1 & \geq & 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq & 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq & 0
\end{array}
$$



System of polynomial inequalities

$$
\begin{array}{lrrrrrrrrr}
A & & -x_1^3 & + x_1 & & + 2x_2 & - 1 & \geq 0 \\
B & -x_2^4 & + 2x_1^2 & - 2x_1x_2 & & + x_2^2 & - \frac{1}{3} & \geq 0 \\
C & & -x_1^2 & - x_2^2 & & + x_1 & + 4 & \geq 0
\end{array}
$$



S

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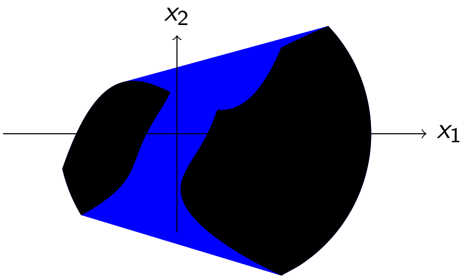
System of polynomial inequalities

$$
\begin{array}{ll}
A & -x_1^3 + x_1 + 2x_2 - 1 \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$



S

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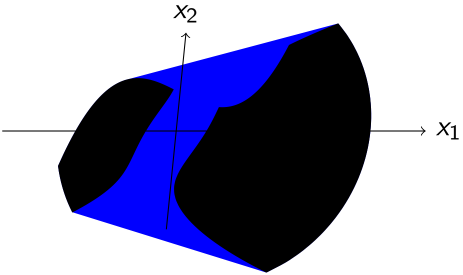
System of polynomial inequalities

$$
\begin{array}{lrrrrrl}
A & & -x_1^3 & +x_1 & +2x_2 & -1 & \geq 0 \\
B & -x_2^4 & +2x_1^2 & -2x_1x_2 & +x_2^2 & -\frac{1}{3} & \geq 0 \\
C & & -x_1^2 & -x_2^2 & +x_1 & +4 & \geq 0
\end{array}
$$



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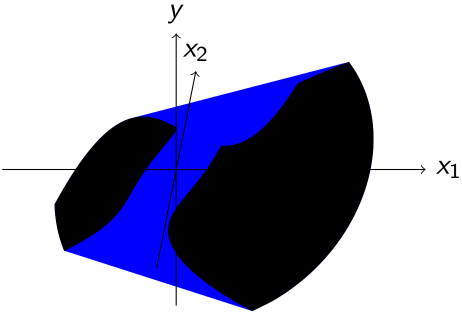
System of polynomial inequalities

$$
\begin{array}{lrrrrrrl}
A & & -x_1^3 & +\ x_1 & +\ 2x_2 & -\ 1 & \geq & 0 \\
B & -x_2^4 & +\ 2x_1^2 & -\ 2x_1x_2 & +\ x_2^2 & -\ \frac{1}{3} & \geq & 0 \\
C & & -x_1^2 & -\ x_2^2 & +\ x_1 & +\ 4 & \geq & 0
\end{array}
$$



S

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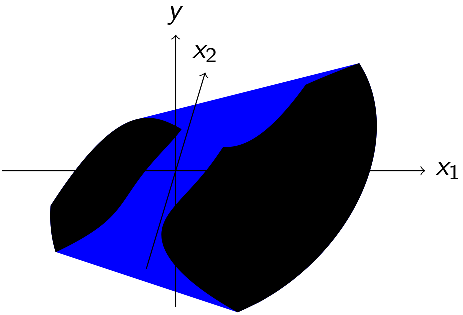
System of polynomial inequalities

$$
\begin{array}{lrrrrrrl}
A & & -x_1^3 & +x_1 & +2x_2 & -1 & \geq & 0 \\
B & -x_2^4 & +2x_1^2 & -2x_1x_2 & +x_2^2 & -\frac{1}{3} & \geq & 0 \\
C & & -x_1^2 & -x_2^2 & +x_1 & +4 & \geq & 0
\end{array}
$$



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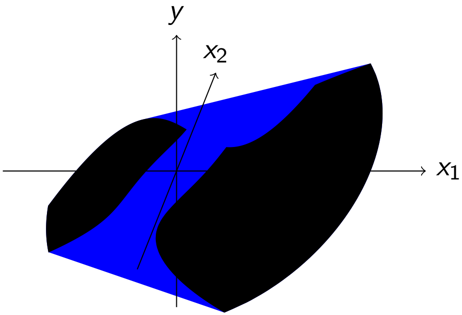
System of polynomial inequalities

$$A \quad -x_1^3 + x_1 + 2x_2 - 1 \geq 0$$

$$B \quad -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \tfrac{1}{3} \geq 0$$

$$C \quad -x_1^2 - x_2^2 + x_1 + 4 \geq 0$$



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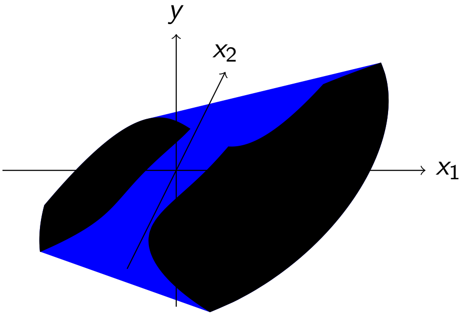
System of polynomial inequalities

$$
\begin{array}{lrrrrrrr}
A & & -\ x_1^3 & +\ x_1 & +\ 2x_2 & -\ 1 & \geq & 0 \\
B & -\ x_2^4 & +\ 2x_1^2 & -\ 2x_1x_2 & +\ x_2^2 & -\ \frac{1}{3} & \geq & 0 \\
C & & -\ x_1^2 & -\ x_2^2 & +\ x_1 & +\ 4 & \geq & 0
\end{array}
$$



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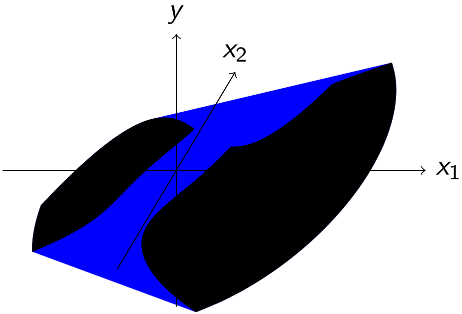
System of polynomial inequalities

A	$-x_1^3 + x_1 + 2x_2 - 1 \geq 0$
B	$-x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0$
C	$-x_1^2 - x_2^2 + x_1 + 4 \geq 0$



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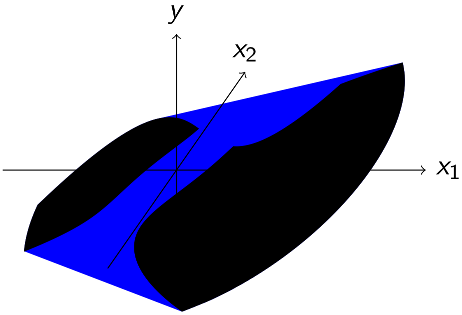
System of polynomial inequalities

$$
\begin{array}{ll}
A & -x_1^3 + x_1 + 2x_2 - 1 \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$



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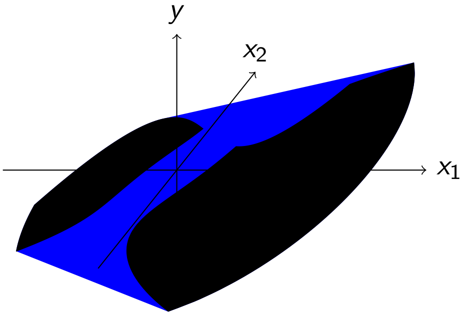
System of polynomial inequalities

A	$-x_1^3 + x_1 + 2x_2 - 1 \geq 0$
B	$-x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0$
C	$-x_1^2 - x_2^2 + x_1 + 4 \geq 0$



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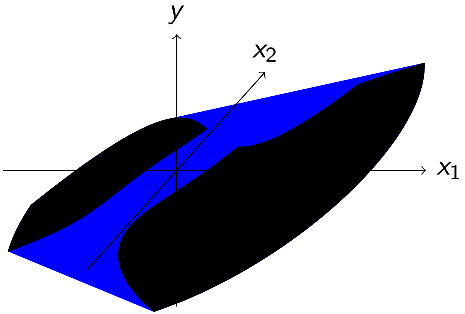
System of polynomial inequalities

$$
\begin{array}{lrl}
A & -x_1^3 + x_1 + 2x_2 - 1 & \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} & \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 & \geq 0
\end{array}
$$



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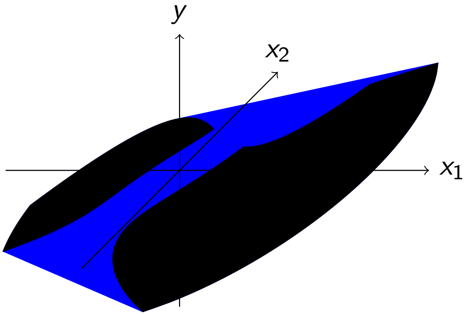
System of polynomial inequalities

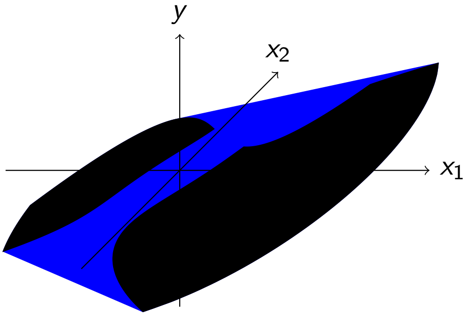
$$
\begin{array}{lrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq 0
\end{array}
$$



conv S

System of polynomial inequalities

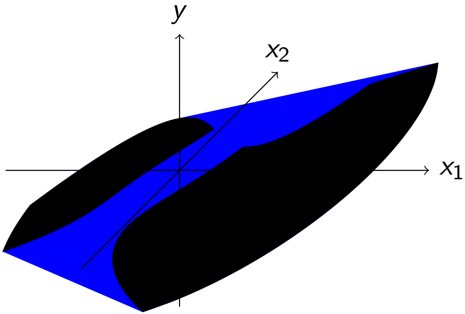
$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

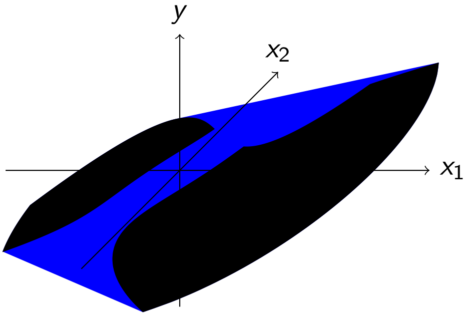
$$
\begin{array}{lrrrrrcl}
A & & -y_1 & +x_1 & +2x_2 & -1 & \geq & 0 \\
B & -x_2^4 & +2x_1^2 & -2x_1x_2 & +x_2^2 & -\frac{1}{3} & \geq & 0 \\
C & & -x_1^2 & -x_2^2 & +x_1 & +4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

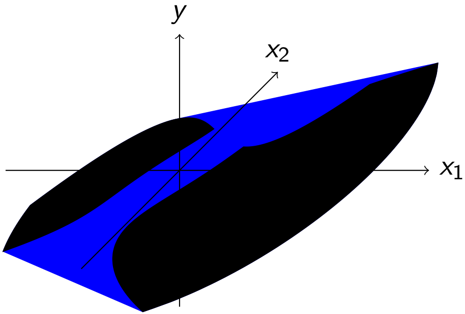
$$
\begin{array}{ll}
A & -y_1 + x_1 + 2x_2 - 1 \geq 0 \\
B & -x_2^4 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0 \\
C & -x_1^2 - x_2^2 + x_1 + 4 \geq 0
\end{array}
$$



conv S

System of polynomial inequalities

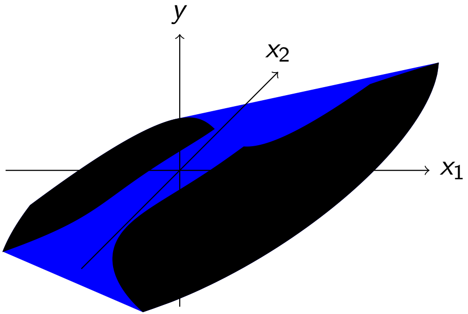
$$\begin{array}{llcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$



conv S

System of polynomial inequalities

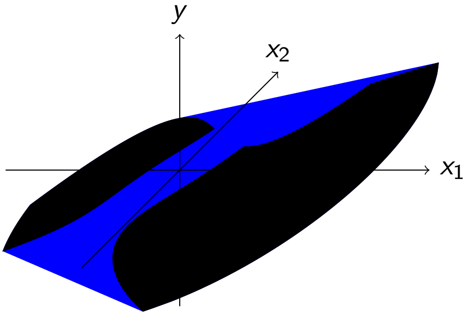
A	$-y_1 + x_1 + 2x_2 - 1 \geq 0$
B	$-y_2 + 2x_1^2 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0$
C	$-x_1^2 - x_2^2 + x_1 + 4 \geq 0$



conv S

System of polynomial inequalities

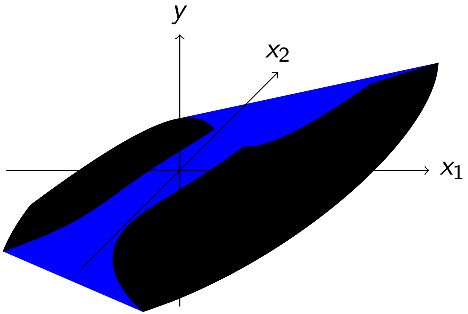
$$
\begin{array}{lrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



conv S

System of polynomial inequalities

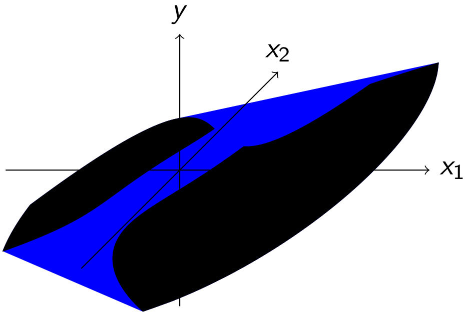
A			$-$	y_1	$+$	x_1	$+$	$2x_2$	$-$	1	$\geq$	0
B	$-$	y_2	$+$	$2y_3$	$-$	$2x_1x_2$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	y_3	$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0



conv S

System of polynomial inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



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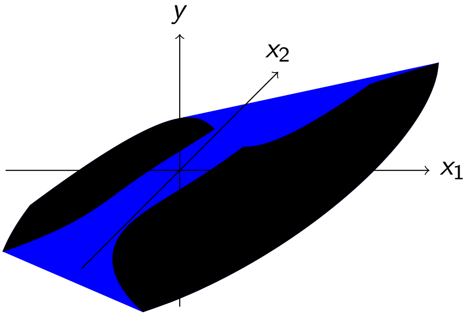
System of polynomial inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$



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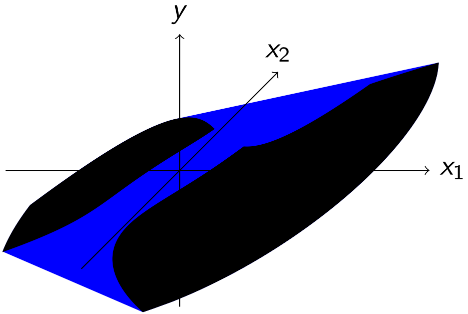
System of linear inequalities

A				$-$	y_1	$+$	x_1	$+$	$2x_2$	$-$	1	$\geq$	0
B	$-$	y_2	$+$	$2y_3$	$-$	$2y_4$	$+$	y_5	$-$	$\frac{1}{3}$	$\geq$	0
C				$-$	y_3	$-$	y_5	$+$	x_1	$+$	4	$\geq$	0



conv S

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrr}
A & & -x_1^3 & +x_1 & +2x_2 & -1 & \geq & 0 \\
B & -x_2^4 & +2x_1^2 & -2x_1x_2 & +x_2^2 & -\frac{1}{3} & \geq & 0 \\
C & & -x_1^2 & -x_2^2 & +x_1 & +4 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant:

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} \\
AB & & x_1^3x_2^4 & - & \ldots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{llllllllllllll}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} \\
AB & & x_1^3x_2^4 & - & \ldots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} & & & & & & & & & & & & \\
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} & & & & & & & & & & & & \\
AB & & x_1^3x_2^4 & - & \ldots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} \\
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrl}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{redundant:} \\
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} & & & & & & & & & & & & \\
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

A			$-$	y_1	$+$	x_1	$+$	$2x_2$	$-$	1	$\geq$	0
B	$-$	x_2^4	$+$	$2x_1^2$	$-$	$2x_1x_2$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	x_1^2	$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0
irredundant:												
AB		$x_1^3x_2^4$	$-$	$\ldots$	$-$	x_2^2	$-$	$\frac{2}{3}x_2$	$+$	$\frac{1}{3}$	$\geq$	0
AC		x_1^5	$+$	$\ldots$	$-$	x_1	$+$	$8x_2$	$-$	4	$\geq$	0
ABC	$-$	$x_1^5x_2^4$	$+$	$\ldots$	$-$	$\frac{13}{3}x_2^2$	$-$	$\frac{8}{3}x_2$	$+$	$\frac{4}{3}$	$\geq$	0
D^2						x_1^2	$-$	$2x_1x_2$	$+$	x_2^2	$\geq$	0
D^2C	$-$	x_1^4	$+$	$\ldots$	$+$	$4x_1^2$	$+$	$4x_1x_2$	$+$	$4x_2^2$	$\geq$	0

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & x_1^3x_2^4 & - & \ldots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & x_1^3x_2^4 & - & \dots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \dots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & x_1^2 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4x_1^2 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

A	$-y_1 + x_1 + 2x_2 - 1 \geq 0$
B	$-y_2 + 2y_3 - 2x_1x_2 + x_2^2 - \frac{1}{3} \geq 0$
C	$-y_3 - x_2^2 + x_1 + 4 \geq 0$
irredundant:	
AB	$x_1^3x_2^4 - \ldots - x_2^2 - \frac{2}{3}x_2 + \frac{1}{3} \geq 0$
AC	$x_1^5 + \ldots - x_1 + 8x_2 - 4 \geq 0$
ABC	$-x_1^5x_2^4 + \ldots - \frac{13}{3}x_2^2 - \frac{8}{3}x_2 + \frac{4}{3} \geq 0$
D^2	$y_3 - 2x_1x_2 + x_2^2 \geq 0$
D^2C	$-x_1^4 + \ldots + 4y_3 + 4x_1x_2 + 4x_2^2 \geq 0$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrl}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & x_1^3x_2^4 & - & \ldots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2x_1x_2 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4x_1x_2 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

A			$-$	y_1	$+$	x_1	$+$	$2x_2$	$-$	1	$\geq$	0
B	$-$	y_2	$+$	$2y_3$	$-$	$2y_4$	$+$	x_2^2	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	y_3	$-$	x_2^2	$+$	x_1	$+$	4	$\geq$	0
irredundant:												
AB		$x_1^3x_2^4$	$-$	$\dots$	$-$	x_2^2	$-$	$\frac{2}{3}x_2$	$+$	$\frac{1}{3}$	$\geq$	0
AC		x_1^5	$+$	$\dots$	$-$	x_1	$+$	$8x_2$	$-$	4	$\geq$	0
ABC	$-$	$x_1^5x_2^4$	$+$	$\dots$	$-$	$\frac{13}{3}x_2^2$	$-$	$\frac{8}{3}x_2$	$+$	$\frac{4}{3}$	$\geq$	0
D^2						y_3	$-$	$2y_4$	$+$	x_2^2	$\geq$	0
D^2C	$-$	x_1^4	$+$	$\dots$	$+$	$4y_3$	$+$	$4y_4$	$+$	$4x_2^2$	$\geq$	0

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrrl}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} & & & & & & & & & & & & \\
AB & & x_1^3x_2^4 & - & \ldots & - & x_2^2 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5x_2^4 & + & \ldots & - & \frac{13}{3}x_2^2 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & x_2^2 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4y_4 & + & 4x_2^2 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & x_1^3 x_2^4 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & x_1^3 x_2^4 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{llrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & y_6 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & x_1^5 & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

A			$-$	y_1	$+$	x_1	$+$	$2x_2$	$-$	1	$\geq$	0
B	$-$	y_2	$+$	$2y_3$	$-$	$2y_4$	$+$	y_5	$-$	$\frac{1}{3}$	$\geq$	0
C			$-$	y_3	$-$	y_5	$+$	x_1	$+$	4	$\geq$	0
irredundant:												
AB		y_6	$-$	$\ldots$	$-$	y_5	$-$	$\frac{2}{3}x_2$	$+$	$\frac{1}{3}$	$\geq$	0
AC		x_1^5	$+$	$\ldots$	$-$	x_1	$+$	$8x_2$	$-$	4	$\geq$	0
ABC	$-$	$x_1^5x_2^4$	$+$	$\ldots$	$-$	$\frac{13}{3}y_5$	$-$	$\frac{8}{3}x_2$	$+$	$\frac{4}{3}$	$\geq$	0
D^2						y_3	$-$	$2y_4$	$+$	y_5	$\geq$	0
D^2C	$-$	x_1^4	$+$	$\ldots$	$+$	$4y_3$	$+$	$4y_4$	$+$	$4y_5$	$\geq$	0

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & y_6 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & y_6 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & x_1^5 x_2^4 & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2 C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & y_6 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & y_{13} & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

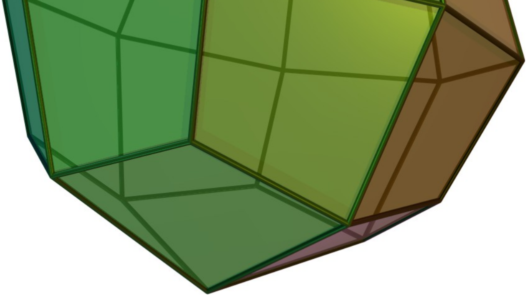
Attempt to linearize after adding redundant inequalities

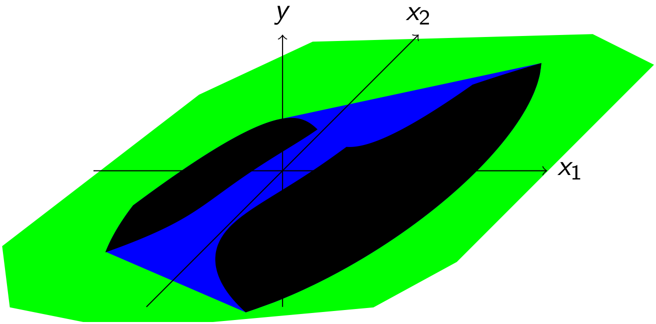
$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} \\
AB & & y_6 & - & \ldots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \ldots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & y_{13} & + & \ldots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & x_1^4 & + & \ldots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0 \\
\text{irredundant:} & & & & & & & & & & & & \\
AB & & y_6 & - & \dots & - & y_5 & - & \frac{2}{3}x_2 & + & \frac{1}{3} & \geq & 0 \\
AC & & y_{10} & + & \dots & - & x_1 & + & 8x_2 & - & 4 & \geq & 0 \\
ABC & - & y_{13} & + & \dots & - & \frac{13}{3}y_5 & - & \frac{8}{3}x_2 & + & \frac{4}{3} & \geq & 0 \\
D^2 & & & & & & y_3 & - & 2y_4 & + & y_5 & \geq & 0 \\
D^2C & - & y_{18} & + & \dots & + & 4y_3 & + & 4y_4 & + & 4y_5 & \geq & 0
\end{array}
$$





conv S

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0 \qquad \iff$$

$$(a+bx_1+cx_2+dx_1^2+ex_1x_2+fx_2^2)\begin{pmatrix}1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2\end{pmatrix}\begin{pmatrix}a\\ b\\ c\\ d\\ e\\ f\end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccc}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2 + dx_1^2 + ex_1x_2 + fx_2^2)^2 \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & x_1^3 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & x_1^3 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\ x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & x_2^4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \\
x_1 & x_1^2 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
x_1^2 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & x_1x_2 & x_2^2 \\ x_1 & y_3 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & x_1x_2 & x_2^2 \\ x_1 & y_3 & x_1x_2 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & x_1x_2 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & x_2^2 \\ x_1 & y_3 & y_4 & y_1 & x_1^2x_2 & x_1x_2^2 \\ x_2 & y_4 & x_2^2 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\ y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\ y_4 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\ x_2^2 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & x_2^2 \\
x_1 & y_3 & y_4 & y_1 & x_1^2 x_2 & x_1 x_2^2 \\
x_2 & y_4 & x_2^2 & x_1^2 x_2 & x_1 x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2 x_2 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & x_1^2 x_2 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
x_2^2 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrl}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & x_1^2 x_2 & x_1 x_2^2 \\
x_2 & y_4 & y_5 & x_1^2 x_2 & x_1 x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2 x_2 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & x_1^2 x_2 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & x_1^2x_2 & x_1x_2^2 \\
x_2 & y_4 & y_5 & x_1^2x_2 & x_1x_2^2 & x_2^3 \\
y_3 & y_1 & x_1^2x_2 & x_1^4 & x_1^3x_2 & x_1^2x_2^2 \\
y_4 & x_1^2x_2 & x_1x_2^2 & x_1^3x_2 & x_1^2x_2^2 & x_1x_2^3 \\
y_5 & x_1x_2^2 & x_2^3 & x_1^2x_2^2 & x_1x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & x_1 x_2^2 \\
x_2 & y_4 & y_5 & y_6 & x_1 x_2^2 & x_2^3 \\
y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & x_1 x_2^2 \\
x_2 & y_4 & y_5 & y_6 & x_1 x_2^2 & x_2^3 \\
y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & x_1 x_2^2 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & x_1 x_2^2 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrl}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\
y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\ y_3 & y_1 & y_6 & x_1^4 & x_1^3 x_2 & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\
y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & x_2^3 \\
y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & x_2^3 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix}
\geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c & d & e & f \end{pmatrix}
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & x_1^3 x_2 & x_1^2 x_2^2 \\
y_4 & y_6 & y_7 & x_1^3 x_2 & x_1^2 x_2^2 & x_1 x_2^3 \\
y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & x_1^2 x_2^2 \\ y_4 & y_6 & y_7 & y_{10} & x_1^2 x_2^2 & x_1 x_2^3 \\ y_5 & y_7 & y_9 & x_1^2 x_2^2 & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & x_1^2x_2^2 \\ y_4 & y_6 & y_7 & y_{10} & x_1^2x_2^2 & x_1x_2^3 \\ y_5 & y_7 & y_9 & x_1^2x_2^2 & x_1x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\ y_4 & y_6 & y_7 & y_{10} & y_{11} & x_1 x_2^3 \\ y_5 & y_7 & y_9 & y_{11} & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\ y_4 & y_6 & y_7 & y_{10} & y_{11} & x_1 x_2^3 \\ y_5 & y_7 & y_9 & y_{11} & x_1 x_2^3 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c & d & e & f \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 & y_3 & y_4 & y_5 \\ x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\ x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\ y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\ y_4 & y_6 & y_7 & y_{10} & y_{11} & y_{12} \\ y_5 & y_7 & y_9 & y_{11} & y_{12} & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix}
1 & x_1 & x_2 & y_3 & y_4 & y_5 \\
x_1 & y_3 & y_4 & y_1 & y_6 & y_7 \\
x_2 & y_4 & y_5 & y_6 & y_7 & y_9 \\
y_3 & y_1 & y_6 & y_8 & y_{10} & y_{11} \\
y_4 & y_6 & y_7 & y_{10} & y_{11} & y_{12} \\
y_5 & y_7 & y_9 & y_{11} & y_{12} & y_2
\end{pmatrix} \succeq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$(-x_1^2 - x_2^2 + x_1 + 4)(a + bx_1 + cx_2) \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{llllllllllllll}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$(-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c \end{pmatrix} (-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$(a + bx_1 + cx_2)^2(-x_1^2 - x_2^2 + x_1 + 4) \geq 0 \qquad \Longleftrightarrow$$

$$\begin{pmatrix} a & b & c \end{pmatrix} (-x_1^2 - x_2^2 + x_1 + 4) \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & x_1^2 & x_1x_2 \\ x_2 & x_1x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \dots & \dots \\
-x_1^3 - x_1x_2^2 + x_1^2 + 4x_1 & \dots & \dots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lccccccccccccc}
A & & & - & x_1^3 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

redundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -x_1^2 - x_2^2 + x_1 + 4 & \ldots & \ldots \\ -x_1^3 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\ -x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \dots & \dots \\
-y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \dots & \dots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & x_2^4 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -x_1^2 - x_2^2 + x_1 + 4 & \ldots & \ldots \\ -y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\ -x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \dots & \dots \\
-y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \dots & \dots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2x_1^2 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & x_1^2 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-x_1^2 - x_2^2 + x_1 + 4 & \ldots & \ldots \\
-y_1 - x_1x_2^2 + x_1^2 + 4x_1 & \ldots & \ldots \\
-x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - x_2^2 + x_1 + 4 & \ldots & \ldots \\ -y_1 - x_1x_2^2 + y_3 + 4x_1 & \ldots & \ldots \\ -x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2x_1x_2 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - x_2^2 + x_1 + 4 & \ldots & \ldots \\ -y_1 - x_1x_2^2 + y_3 + 4x_1 & \ldots & \ldots \\ -x_1^2x_2 - x_2^3 + x_1x_2 + 4x_2 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - x_2^2 + x_1 + 4 & \ldots & \ldots \\
-y_1 - x_1 x_2^2 + y_3 + 4x_1 & \ldots & \ldots \\
-x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & x_2^2 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & x_2^2 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - x_2^2 + x_1 + 4 & \ldots & \ldots \\ -y_1 - x_1 x_2^2 + y_3 + 4x_1 & \ldots & \ldots \\ -x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - y_5 + x_1 + 4 & \ldots & \ldots \\
-y_1 - x_1 x_2^2 + y_3 + 4x_1 & \ldots & \ldots \\
-x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - y_5 + x_1 + 4 & \dots & \dots \\
-y_1 - x_1 x_2^2 + y_3 + 4x_1 & \dots & \dots \\
-x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - y_5 + x_1 + 4 & \ldots & \ldots \\
-y_1 - y_6 + y_3 + 4x_1 & \ldots & \ldots \\
-x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lrrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - y_5 + x_1 + 4 & \ldots & \ldots \\ -y_1 - y_6 + y_3 + 4x_1 & \ldots & \ldots \\ -x_1^2 x_2 - x_2^3 + y_4 + 4x_2 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - y_5 + x_1 + 4 & \ldots & \ldots \\ -y_1 - y_6 + y_3 + 4x_1 & \ldots & \ldots \\ -y_7 - x_2^3 + y_4 + 4x_2 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lrrrrrrrrrrrrrr}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix}
\begin{pmatrix}
-y_3 - y_5 + x_1 + 4 & \dots & \dots \\
-y_1 - y_6 + y_3 + 4x_1 & \dots & \dots \\
-y_7 - x_2^3 + y_4 + 4x_2 & \dots & \dots
\end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

System of polynomial inequalities

Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \dots$):

$$
\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -y_3 - y_5 + x_1 + 4 & \dots & \dots \\ -y_1 - y_6 + y_3 + 4x_1 & \dots & \dots \\ -y_7 - y_8 + y_4 + 4x_2 & \dots & \dots \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0
$$

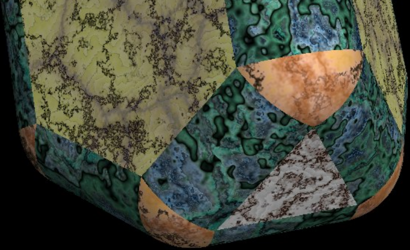
System of polynomial inequalities

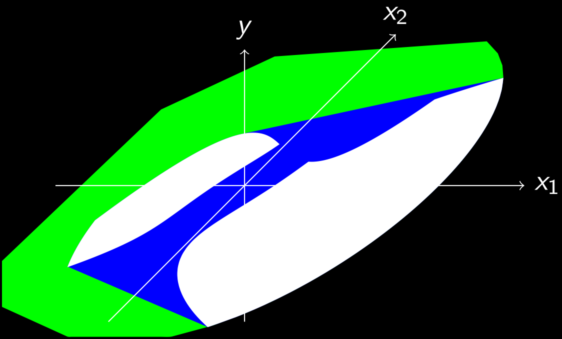
Attempt to linearize after adding families of redundant inequalities

$$
\begin{array}{lcccccccccccccc}
A & & & - & y_1 & + & x_1 & + & 2x_2 & - & 1 & \geq & 0 \\
B & - & y_2 & + & 2y_3 & - & 2y_4 & + & y_5 & - & \frac{1}{3} & \geq & 0 \\
C & & & - & y_3 & - & y_5 & + & x_1 & + & 4 & \geq & 0
\end{array}
$$

irredundant families (parametrized by $a, b, c, \ldots$):

$$
\begin{pmatrix}
-y_3 - y_5 + x_1 + 4 & \ldots & \ldots \\
-y_1 - y_6 + y_3 + 4x_1 & \ldots & \ldots \\
-y_7 - y_8 + y_4 + 4x_2 & \ldots & \ldots
\end{pmatrix} \qquad \succeq 0
$$





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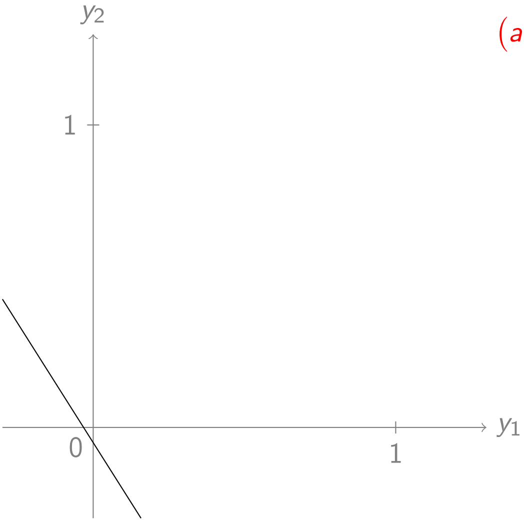
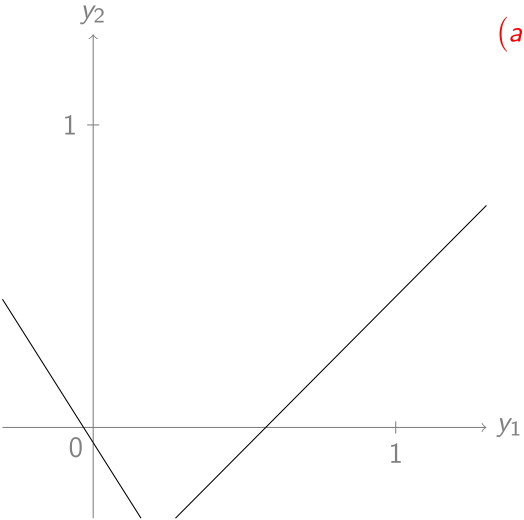
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

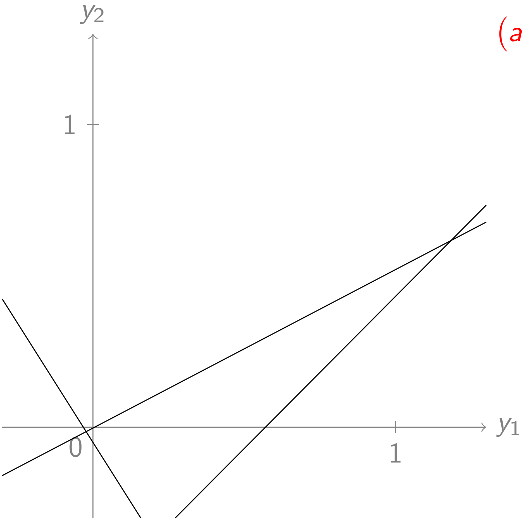
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

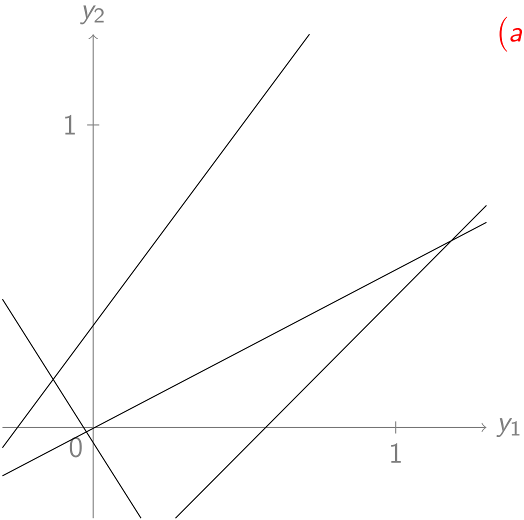
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

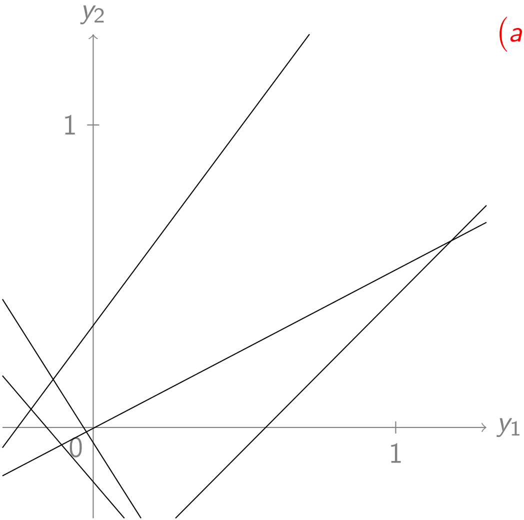
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

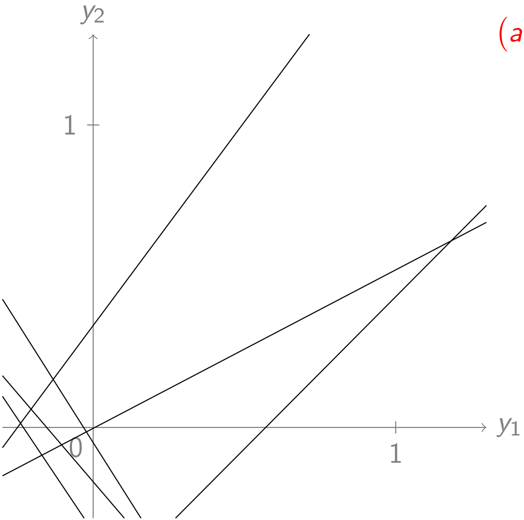
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

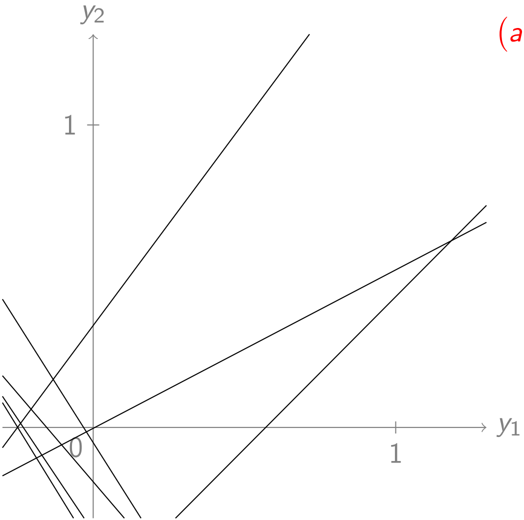
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

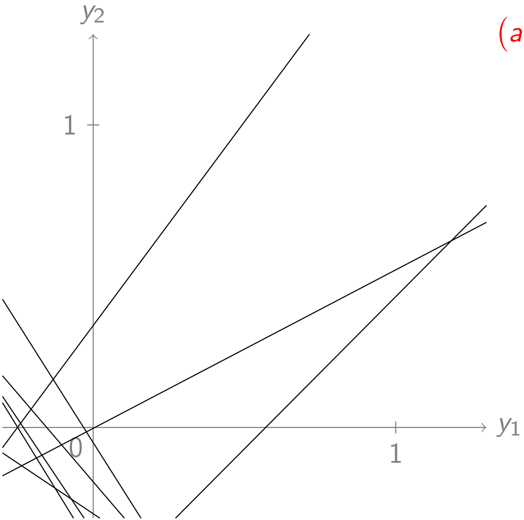
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

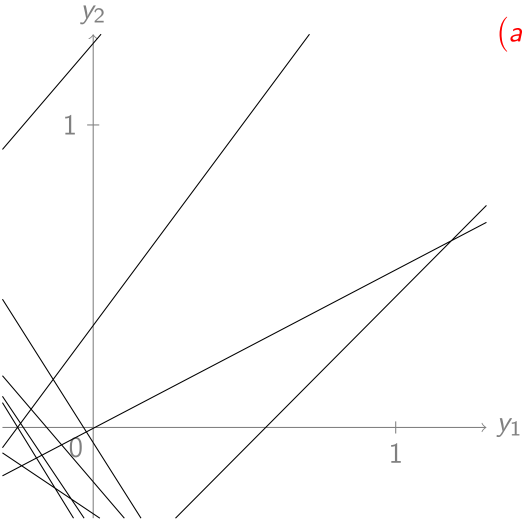
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

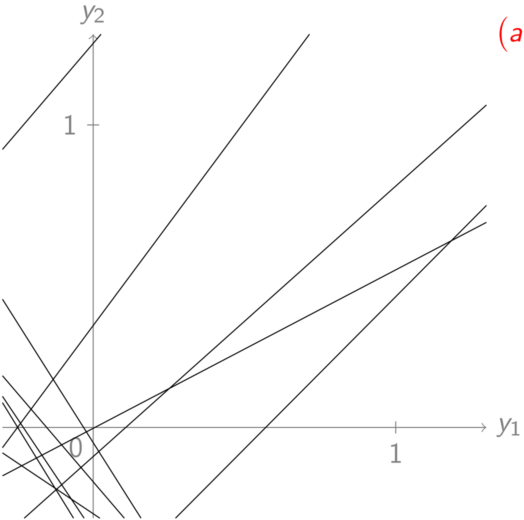
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

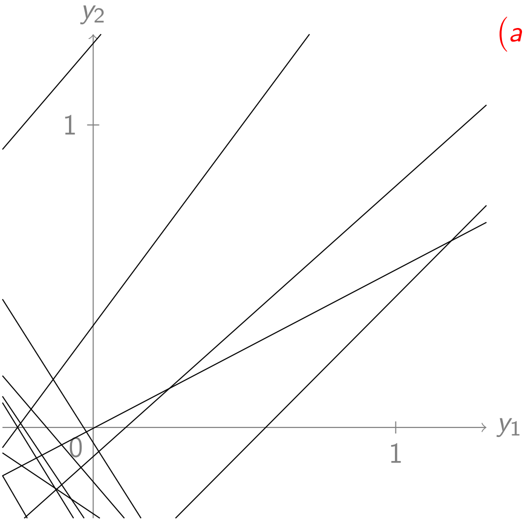
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

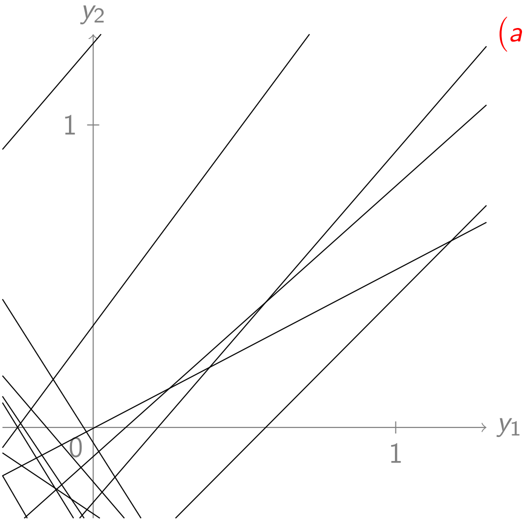
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

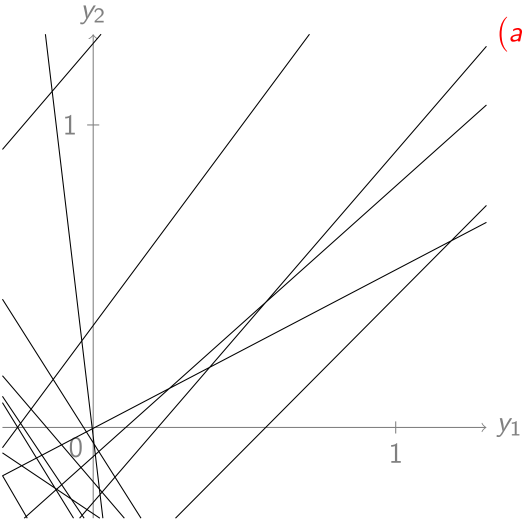
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

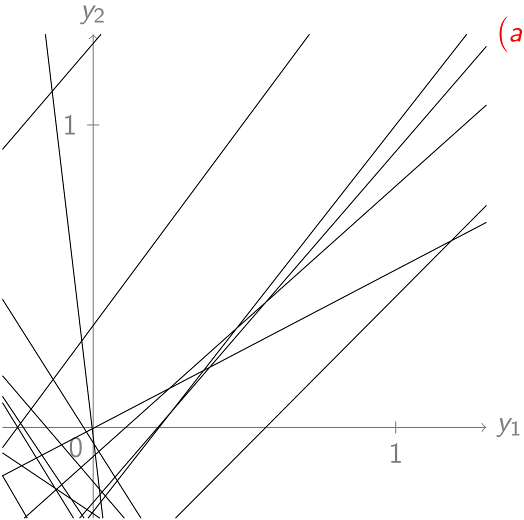
Linear matrix inequality



$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} y_1 & x_2 & y_1 \\ y_2 & 1 & y_1 \\ y_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \geq 0$$

a, b, c independant
and normally distributed

Linear matrix inequality



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a, b, c independant
and normally distributed

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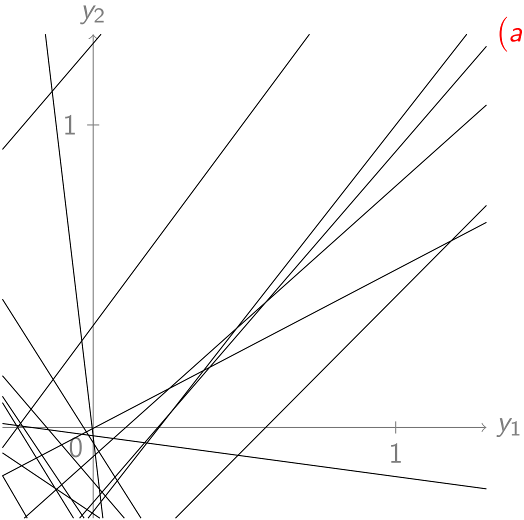

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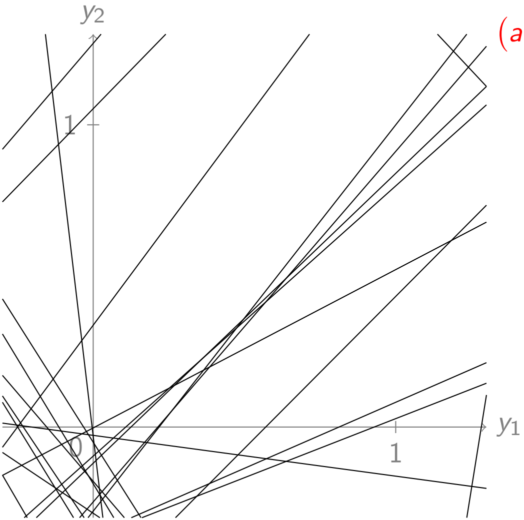

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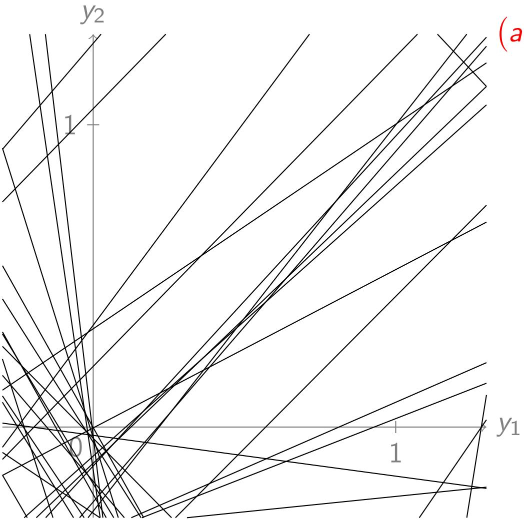

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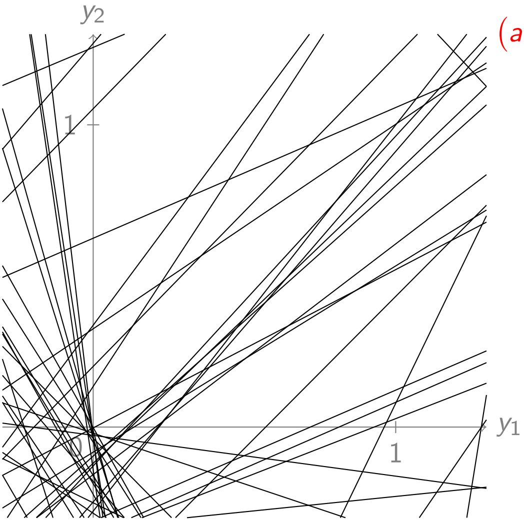

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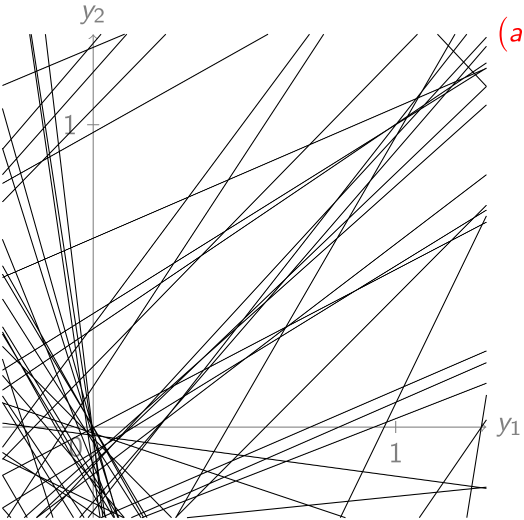

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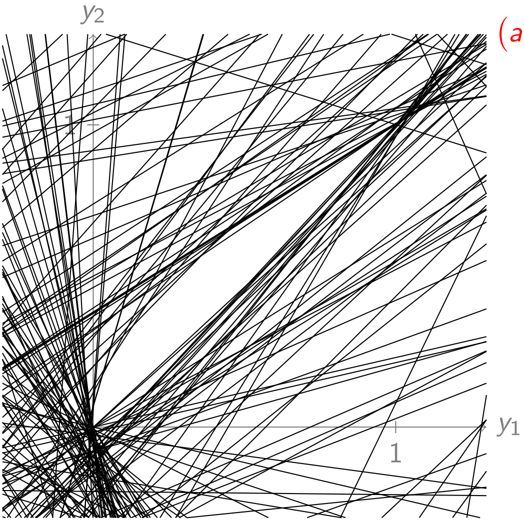

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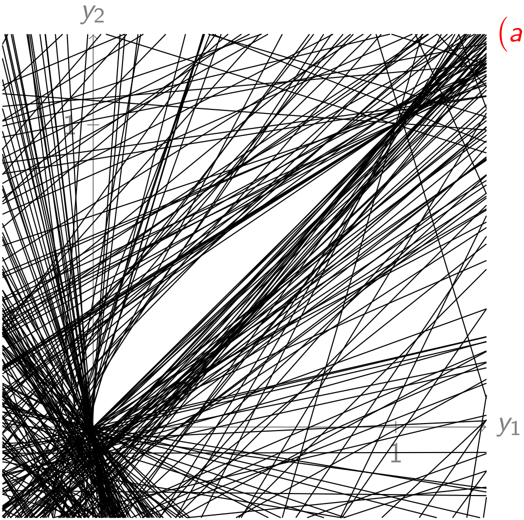

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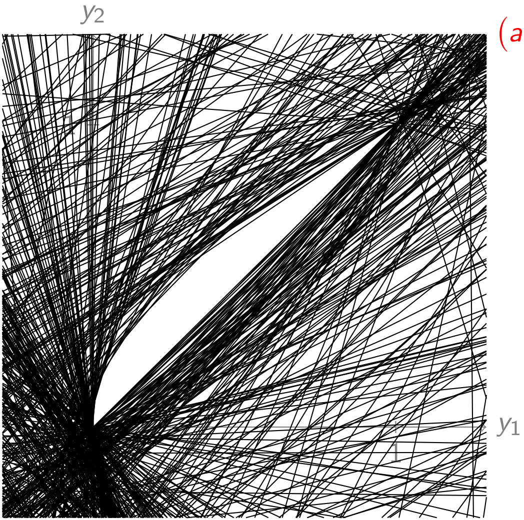

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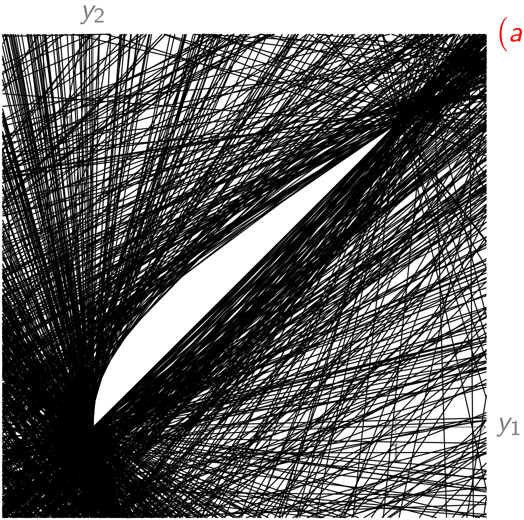

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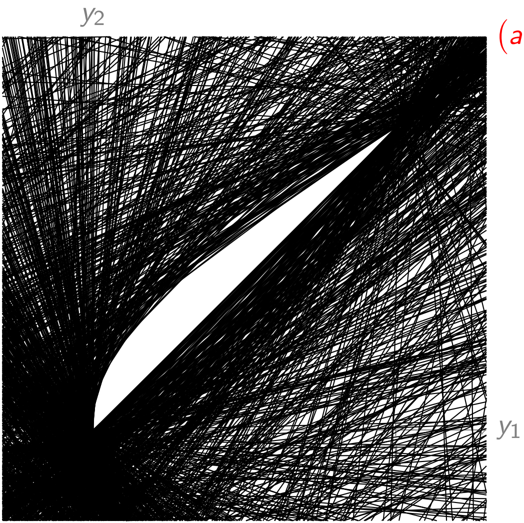

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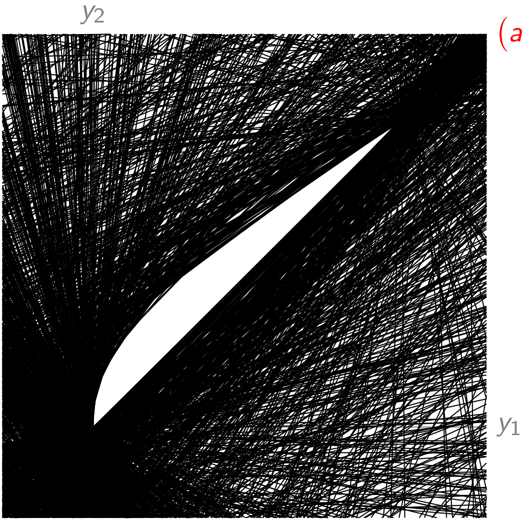

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If S is non-compact, then often $\operatorname{conv} S \neq S'$ and hence the answer is often no. If S is compact, then we will see that $\operatorname{conv} S = S'$ but Parrilo gave in his 2006 Banff talk an example where the answer nevertheless is no.

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Proposition. Fix $k \in \mathbb{N} := \{1, 2, 3, \dots\}$ and suppose $S \neq \emptyset$ and $\operatorname{conv} S$ is closed (e.g. S is compact). Then the following are equivalent:

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Proposition (Powers & Scheiderer 2005).
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Corollary. If S is compact, then $\operatorname{conv} S = S'$.

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Problem: Dependance on $\frac{\|f\|}{f^*}$.

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Problem: We do not get the necessary degree bounds in this way.

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Example. $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} \colon \left(\begin{smallmatrix} x & 1 \\ 1 & y \end{smallmatrix}\right) \succeq 0\}$ is an LMI projection.

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Nemirovski asked in the ICM in Madrid 2006 whether any convex semialgebraic set is an LMI projection: "This question seems to be completely open."

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