Using semidefinite programming for polynomial optimization problems

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Self-dual convex cones

Definition. A subset $K \subseteq E$ of a real vector space E is called a convex cone if $0 \in K$, $K + K \subseteq K$ and $\mathbb{R}_{\geq 0} K \subseteq K$.

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- $E = \mathbb{R}^n$, $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$, $K = (\mathbb{R}_{\geq 0})^n$
- $E = S\mathbb{R}^{n \times n}$ (symmetric $n \times n$ matrices),
  $\langle A, B \rangle = \sum_{i,j=1}^n A_{ij} B_{ij} = \mathrm{tr}(AB^T) = \mathrm{tr}(AB)$,
  $K = S\mathbb{R}_+^{n \times n}$ (psd, positive semidefinite)

Matrix scalar products

- Regard the Euclidean space $\mathbb{R}^{m\times n}$ of all $m \times n$ matrices with

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Then for all matrices A, B, C such that $\langle AB, C\rangle$ is defined,

$$\langle AB, C\rangle = \mathrm{tr}(ABC^T) = \mathrm{tr}(BC^TA) = \mathrm{tr}(B(A^TC)^T) = \langle B, A^TC\rangle,$$

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$$\|A\| = \|\lambda(A)\|$$

where $\lambda(A)$ is the diagonal of D containing the eigenvalues of A.

Some descriptions of the cone $S\mathbb{R}_+^{n\times n}$

Proposition: For any matrix $A \in S\mathbb{R}^{n\times n}$ are equivalent:

(i) A is positive semidefinite.

(ii) $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$

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(viii) $\langle A, B\rangle \geq 0$ for all $B \in S\mathbb{R}_+^{n\times n}$. (shows self-duality)

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- Most of the concepts for linear programming can be adapted to semidefinite programming.
- In a certain sense (not restrictive in practice), semidefinite programming is solvable in polynomial time.
- A lot of efficient semidefinite programming solvers are freely available.

Programming over self-dual cones

Let E, F be finite-dimensional Euclidean spaces,
$K \subseteq E$ a self-dual convex cone, $c \in E$, $b \in F$,
$\mathcal{A} : E \to F$ a linear map and $\mathcal{A}^* : F \to E$ its adjoint.

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\begin{array}{lll}
(P) & \text{minimize} & \langle c, x \rangle \\
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(D) & \text{maximize} & \mu \\
 & \text{subject to} & \mu \in \mathbb{R} \\
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Weak duality: If x is feasible for (P) and y for (D), then

$$\langle c, x \rangle \geq \langle \mathcal{A}^* y, x \rangle = \langle \mathcal{A}x, y \rangle = \langle b, y \rangle.$$

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Write $P^* := \inf(P) := \inf\{\langle c, x \rangle \mid x \in K, \mathcal{A}x = b\} \in \mathbb{R} \cup \{\pm\infty\}$ and (analogously) $D^* := \sup(D)$ for the optimal values of (P) and (D).

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Strong duality $P^* = D^*$ holds often,

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Weak duality: $P^* \geq D^*$
Strong duality $P^* = D^*$ holds often, for example if both problems are feasible and one of them strictly, i.e., with K replaced by its interior.

Semidefinite Programming

Let $A_1, \ldots, A_m \in S\mathbb{R}^{n \times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{n \times n}$,
$\mathcal{A} : S\mathbb{R}^{n \times n} \to \mathbb{R}^m : X \mapsto (\langle A_i, X \rangle)_{i \in \{1,\ldots,m\}}$.

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$\mathcal{A}^* : \mathbb{R}^m \to S\mathbb{R}^{n \times n} : y \mapsto \sum_{i=1}^m y_i A_i$ since

$$\langle \mathcal{A}X, y \rangle = \sum_{i=1}^m \langle A_i, X \rangle y_i = \sum_{i=1}^m y_i \langle X, A_i \rangle = \langle X, \sum_{i=1}^m y_i A_i \rangle.$$

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(P)	minimize	$\langle C, X\rangle$	(D)	maximize	$\langle b, y\rangle$
	subject to	$X \in S\mathbb{R}_+^{n\times n}$		subject to	$y \in \mathbb{R}^m$
		$\mathcal{A}X = b$			$C - \mathcal{A}^* y$ psd

Semidefinite Programming

Let $A_1, \ldots, A_m \in S\mathbb{R}^{n\times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{n\times n}$,
$\mathcal{A} : S\mathbb{R}^{n\times n} \to \mathbb{R}^m : X \mapsto (\langle A_i, X\rangle)_{i\in\{1,\ldots,m\}}$. Then
$\mathcal{A}^* : \mathbb{R}^m \to S\mathbb{R}^{n\times n} : y \mapsto \sum_{i=1}^m y_i A_i$ since

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Weak duality: $P^* \geq D^*$
Strong duality $P^* = D^*$ holds often, for example if both problems are feasible and one of them strictly, i.e., with “psd” replaced by “pd”.

Positive semidefinite matrices and families of vectors

Recall the following fact.

A real symmetric $n \times n$ matrix A is psd if and only if there are vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ such that

$$A = \begin{pmatrix} \langle v_1, v_1 \rangle & \ldots & \langle v_1, v_n \rangle \\ \vdots & & \vdots \\ \langle v_n, v_1 \rangle & \ldots & \langle v_n, v_n \rangle \end{pmatrix}.$$

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Therefore SDP can be seen as optimization over families of vectors where the goal function and the constraints are linear in the scalar products between these vectors.

The maximum cut problem

Given a graph, i.e., an $n \in \mathbb{N}$ (number of nodes) and a set

$$E \subseteq \{(i, j) \in \{1, \ldots, n\}^2 \mid i < j\}$$

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$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{(i,j)\in E} \frac{1}{2}(1 - x_i x_j) \\ \text{subject to} & x_i \in \mathbb{R} \text{ for all } i \in \{1,\dots,n\} \\ & x_i^2 = 1 \end{array}$$

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Vector version of first MAXCUT relaxation

$$(P_1) \qquad \text{maximize} \quad \sum_{(i,j)\in E} \frac{1}{2}(1 - \langle v_i, v_j \rangle)$$

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Error analysis of Goemans & Williamson: Computing an optimal solution $v_1, \ldots, v_n \in S^{n-1}$

J. Assoc. Comput. Mach. 42, No.6, 1115–1145 (1995)

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Note: With obvious changes, one can allow affine linear goal functions. From now on, it will be more efficient to implement all our primals as duals and vice versa.

What is the dual of the first relaxation?

An exercise shows that solving the dual SDP (D_1) amounts to minimizing $\mu \in \mathbb{R}$ subject to the following constraint:

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Obviously, there is no duality gap between (P_1) and (D_1).

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Exactness of the n-th MAXCUT relaxation

Proposition. Suppose $p \in \mathbb{R}[X_1, \ldots, X_n]$ such that

$$p \geq 0 \text{ on } \{-1, 1\}^n.$$

Then f is a square modulo the ideal

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Proof by algebra. By chinese remainder theorem

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Corollary. $D_n^* = P_n^* = f^*$

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- ...the set $S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$

n

f

$g_1, \ldots, g_m$

S

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We consider the problem of minimizing f on S.

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and, if possible, a minimizer, i.e., an element of the set

$$S^* := \{x^* \in S \mid f(x^*) \leq f(x) \text{ for all } x \in S\}.$$

L **P**

Linear Programming

$$\text{minimize} \quad f(x)$$

$$\text{subject to} \quad x \in \mathbb{R}^n$$

$$g_1(x) \geq 0$$

$$\vdots$$

$$g_m(x) \geq 0$$

where all polynomials f and g_i are linear, i.e., their degree is ≤ 1. In particular, $S \subseteq \mathbb{R}^n$ is a polyhedron.

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S **D** **P**

$$\begin{array}{rl} \text{minimize} & f(x) \\ \\ \text{subject to} & x \in \mathbb{R}^n \\ & \begin{pmatrix} g_{11}(x) & \dots & g_{1m}(x) \\ \vdots & \ddots & \vdots \\ & \dots & g_{mm}(x) \end{pmatrix} \text{ is psd} \end{array}$$

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Semidefinite Programming

$$
\begin{array}{rl}
\text{minimize} & f(x) \\
\\
\text{subject to} & x \in \mathbb{R}^n \\
& \begin{pmatrix} g_{11}(x) & \dots & g_{1m}(x) \\ \vdots & \ddots & \vdots \\ & \dots & g_{mm}(x) \end{pmatrix} \text{is psd}
\end{array}
$$

where all polynomials f and g_{ij} are linear, i.e., their degree is ≤ 1.

Duality

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Duality

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$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=0}^{2d} a_i x^i \\ \text{subject to} & x \in \mathbb{R} \end{array}$$

where $a_0, \ldots, a_{2d} \in \mathbb{R}$.

$$\text{minimize} \quad \sum_{i=0}^{2d} a_i x^i$$

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Note that $\begin{pmatrix} 1 & x & x^2 & \dots & x^d \\ x & x^2 & \ddots & \ddots & \\ x^2 & \ddots & \ddots & & \\ \vdots & \ddots & & & \\ x^d & & & & x^{2d} \end{pmatrix}$ is psd

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where $a_0, \dots, a_{2d} \in \mathbb{R}$.

$$(P) \qquad \text{minimize} \quad \sum_{i=1}^{2d} a_i y_i + a_0$$

$$\text{subject to} \quad y \in \mathbb{R}^{2d}$$

$$\begin{array}{c} \\ 1 \\ X \\ X^2 \\ \vdots \\ X^d \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & X & X^2 & \dots & X^d \end{array} \\ \begin{pmatrix} 1 & y_1 & y_2 & & y_d \\ y_1 & y_2 & \ddots & \ddots & \\ y_2 & \ddots & \ddots & & \\ \vdots & \ddots & & & \\ y_d & & & & y_{2d} \end{pmatrix} \end{array} \text{ is psd}$$

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Proposition. For every $p \in \mathbb{R}[X]$,

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Corollary.

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Theorem (Hilbert). For every $p \in \mathbb{R}[X, Y]$ of degree ≤ 4,

$$p \geq 0 \text{ on } \mathbb{R}^2 \implies p \text{ is a sum of three squares in } \mathbb{R}[X, Y].$$

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The Motzkin polynomial

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- Described method always yields certified lower bounds, but they might by $-\infty$:

$$-\infty \leq D^* = P^* \leq f^*$$

- But there are a lot of remedies...

Case where S is compact.

For simplicity, we suppose $m = 1$ and write $g := g_1$ (technical difficulties which are however not very serious otherwise), i.e.

$$S = \{x \in \mathbb{R}^n \mid g(x) \geq 0\}.$$

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We will later present in detail Lasserre's method which produces now a sequence $(P_k)_{2k \geq d}$ of relaxations such that

$$D_k^* \leq P_k^* \leq f^* \qquad \text{and} \qquad \lim_{k \to \infty} D_k^* = \lim_{k \to \infty} P_k^* = f^*.$$

$$\text{minimize} \quad \sum_{|\alpha| \leq d} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

$$\text{subject to} \quad x \in S$$

where $k \in \mathbb{N}$, $2k \geq d$, $a_\alpha \in \mathbb{R}$ $(|\alpha| \leq k)$.

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Note that

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$$\text{subject to} \quad y_\alpha \in \mathbb{R} \qquad (|\alpha| \le k)$$

$$\begin{matrix} & 1 & X_1 & \dots & X_n^k \\ 1 & & & & \\ X_1 & & & & \\ \vdots & & & & \\ X_n^k & & & & \end{matrix} \left(\begin{matrix} \begin{pmatrix} 1 & y_{10\dots0} & \cdots & \\ y_{10\dots0} & & & \\ \vdots & & & \\ & & & \end{pmatrix} & \\ & \begin{pmatrix} \text{“localization} \\ \text{matrix”} \end{pmatrix} \end{matrix} \right) \text{ is psd}$$

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Implementations

- Henrion, Lasserre: GloptiPoly
  `http://www.laas.fr/~henrion/software/gloptipoly/`
- Loefberg: YALMIP
  `http://control.ee.ethz.ch/~joloef/yalmip.php`
- Prajna, Papachristodoulou, Seiler, Parrilo: SOSTOOLS
  `http://www.cds.caltech.edu/sostools/`
- Waki, Kim, Kojima, Muramatsu: SparsePOP
  `http://www.is.titech.ac.jp/~kojima/SparsePOP/`
- All run under Matlab.
- All run with the free SeDuMi solver by Jos Sturm.
- Some support other solvers, too.

Lasserre's hierarchy of relaxations

for optimization of polynomials on
compact basic closed semialgebraic sets

Notation

- $\bar{X} := (X_1, \dots, X_n)$ variables
- $\mathbb{R}[\bar{X}] := \mathbb{R}[X_1, \dots, X_n]$ polynomial ring
- $f \in \mathbb{R}[\bar{X}]$ an arbitrary polynomial
- $g_1, \dots, g_m \in \mathbb{R}[\bar{X}]$ polynomials defining...
- ... the set $S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$
- $g_0 := 1 \in \mathbb{R}[\bar{X}]$ for convenience
- $M := \sum_{i=0}^m \sum \mathbb{R}[\bar{X}]^2 g_i = \left\{\sum_{i=0}^m \sigma_i g_i \mid \sigma_i \in \sum \mathbb{R}[\bar{X}]^2\right\}$
  the quadratic module generated by $g_1, \dots, g_m$

n

f

$g_1, \ldots, g_m$

S

g_0

M

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$$N - \sum_{i=1}^{n} X_i^2 \in M$$

for some $N \in \mathbb{N}$.

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In particular, S is compact.

Optimization

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$$f^* := \inf\{f(x) \mid x \in S\} \in \mathbb{R} \cup \{\infty\}$$

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$$f^* := \inf\{f(x) \mid x \in S\} \in \mathbb{R} \cup \{\infty\}$$

and, if possible, a minimizer, i.e., an element of the set

$$S^* := \{x^* \in S \mid \forall x \in S : f(x^*) \leq f(x)\}.$$

Convexification

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$$f^* = \inf \left\{ \int f d\mu \mid \mu \in \mathcal{M}^1(S) \right\}$$

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- Take a dual standpoint:

$$f^* = \sup\{a \in \mathbb{R} \mid f - a \geq 0 \text{ on } S\} = \sup\{a \in \mathbb{R} \mid f - a > 0 \text{ on } S\}$$

Describing measures and positive polynomials

Putinar's solution to the moment problem. For every map $L : \mathbb{R}[\bar{X}] \to \mathbb{R}$ are equivalent:

(1) L is linear, $L(1) = 1$ and $L(M) \subseteq \mathbb{R}_{\geq 0}$

(2) $\exists \mu \in \mathcal{M}^1(S) : \forall p \in \mathbb{R}[\bar{X}] : L(p) = \displaystyle\int p d\mu$

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Putinar's Positivstellensatz. $f > 0$ on $S \implies f \in M$

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$\mathbb{R}[\bar{X}]$ polynomial ring

$$M := \sum_{i=0}^{m} \sum \mathbb{R}[\bar{X}]^2 \; g_i \qquad \text{quadratic module}$$

$$= \left\{ \sum_{i=0}^{m} \sigma_i g_i \mid \sigma_i \in \sum \mathbb{R}[\bar{X}]^2 \right\}$$

Introduce finite-dimensional analogues $M_k \subseteq \mathbb{R}[\bar{X}]_k$ of $M \subseteq \mathbb{R}[\bar{X}]$.

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for arbitrary

$k \in \mathcal{N} := \{s \in \mathbb{N} \mid s \geq \max\{\deg g_0, \ldots, \deg g_m, \deg f\}\}$.

Introduce finite-dimensional analogues $M_k \subseteq \mathbb{R}[\bar{X}]_k$ of $M \subseteq \mathbb{R}[\bar{X}]$.

$$\mathbb{R}[\bar{X}]_k := \{p \mid p \in \mathbb{R}[\bar{X}], \deg p \leq k\} \qquad \text{real vector space}$$

$$M_k := \sum_{i=0}^{m} \sum \mathbb{R}[\bar{X}]_{d_i}^2 g_i \qquad \text{convex cone}$$

$$= \left\{ \sum_{i=0}^{m} \sigma_i g_i \mid \sigma_i \in \sum \mathbb{R}[\bar{X}]^2, \deg(\sigma_i g_i) \leq k \right\}$$

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Here $d_i := \max\{e \in \mathbb{N} \mid 2e + \deg g_i \leq k\}$.

Warning: Never confuse M_k with $M \cap \mathbb{R}[\bar{X}]_k \supseteq M_k$.

We saw that

$$f^* = \inf\{L(f) \mid L : \mathbb{R}[\bar{X}] \to \mathbb{R} \text{ is linear}, L(1) = 1, L(M) \subseteq \mathbb{R}_{\geq 0}\} \qquad \text{and}$$

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In analogy to this, we set

$$P_k^* = \inf\{L(f) \mid L : \mathbb{R}[\bar{X}]_k \to \mathbb{R} \text{ is linear}, L(1) = 1, L(M_k) \subseteq \mathbb{R}_{\geq 0}\} \quad \text{and}$$

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for every $k \in \mathcal{N}$.

$P_k^* \in \mathbb{R} \cup \{\pm\infty\}$ and $D_k^* \in \mathbb{R} \cup \{\pm\infty\}$ are the optimal values of the following pair of optimization problems...

$$
\begin{array}{llll}
(P_k) & \text{minimize} & L(f) & \text{subject to} & L : \mathbb{R}[\bar{X}]_k \to \mathbb{R} \text{ is linear,} \\
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Theorem (Lasserre). $(D_k^*)_{k \in \mathcal{N}}$ and $(P_k^*)_{k \in \mathcal{N}}$ are increasing sequences that converge to f^* and satisfy $D_k^* \leq P_k^* \leq f^*$.

Proof.

$$
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(P_k) & \text{minimize} & L(f) & \text{subject to} & L : \mathbb{R}[\bar{X}]_k \to \mathbb{R} \text{ is linear,} \\
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Clear: $(P_k^*)_{k \in \mathbb{N}}$ and $(D_k^*)_{k \in \mathbb{N}}$ increase.

$$
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Convergence of $(D_k^*)_{k \in \mathbb{N}}$ implies convergence of $(P_k^*)_{k \in \mathbb{N}}$.

$$
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(P_k)	minimize	$L(f)$	subject to	$L : \mathbb{R}[\bar{X}]_k \to \mathbb{R}$ is linear,
k–th primal relaxation				$L(1) = 1$ and
(primal relaxation of order k)				$L(M_k) \subseteq \mathbb{R}_{\geq 0}$
(D_k)	maximize	a	subject to	$a \in \mathbb{R}$ and
k–th dual relaxation				$f - a \in M_k$

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(P_k) and (D_k) can be formulated as a primal-dual pair of semidefinite programs.

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Jean Lasserre: Global optimization with polynomials and the problem of moments
SIAM J. Optim. **11**, No. 3, 796–817 (2001)

$$
\begin{array}{llll}
(P_k) & \text{minimize} & L(f) & \text{subject to} \quad L : \mathbb{R}[\bar{X}]_k \to \mathbb{R} \text{ is linear,} \\
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 & & & \phantom{\text{subject to}} \quad L(M_k) \subseteq \mathbb{R}_{\geq 0} \\
(D_k) & \text{maximize} & a & \text{subject to} \quad a \in \mathbb{R} \text{ and} \\
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Theorem (Lasserre). $(D_k^*)_{k \in \mathcal{N}}$ and $(P_k^*)_{k \in \mathcal{N}}$ are increasing sequences that converge to f^* and satisfy $D_k^* \leq P_k^* \leq f^*$. How fast?

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Theorem. Suppose $m = 1$ and $g := g_1$. Then there exists $C \in \mathbb{N}$ depending on f and g and $c \in \mathbb{N}$ depending on g such that

$$f^* - D_k^* \leq \frac{C}{\sqrt[c]{k}} \qquad \text{for big } k.$$

On the complexity of Schmüdgen's Positivstellensatz
Journal of Complexity **20**, No. 4, 529—543 (2004)

$$
\begin{array}{llll}
(P_k) & \text{minimize} & L(f) & \text{subject to} \quad L : \mathbb{R}[\bar{X}]_k \to \mathbb{R} \text{ is linear,} \\
& & & \phantom{\text{subject to} \quad} L(1) = 1 \text{ and} \\
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(D_k) & \text{maximize} & a & \text{subject to} \quad a \in \mathbb{R} \text{ and} \\
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$$
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$$

Dependance on f can be made explicit. Proof hints to make dependance on g explicit for concrete g.

$$
\begin{array}{llll}
(P_k) & \text{minimize} & L(f) & \text{subject to} \quad L : \mathbb{R}[\bar{X}]_k \to \mathbb{R} \text{ is linear,} \\
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$$
f^* - D_k^* \leq \frac{C}{\sqrt[c]{k}} \qquad \text{for big } k.
$$

In practice: Convergence usually very fast,
often $D_k^* = P_k^* = f^*$ for small k.

$$
\begin{array}{llll}
(P_k) & \text{minimize} & L(f) & \text{subject to} & L : \mathbb{R}[\bar{X}]_k \to \mathbb{R} \text{ is linear,} \\
& & & & L(1) = 1 \text{ and} \\
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Putinar's Positivstellensatz implies convergence of $(D_k^*)_{k \in \mathcal{N}}$ and therefore of $(P_k^*)_{k \in \mathcal{N}}$.

What can we know from Putinar's solution to the moment problem?

$$
\begin{array}{llll}
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A priori nothing!

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What can we know from Putinar's solution to the moment problem?
A priori nothing! But with additional compactness arguments involving Tychonoff's Theorem, the following...

$$(P_k) \quad \text{minimize} \quad L(f) \quad \text{subject to} \quad \begin{aligned} & L : \mathbb{R}[\bar{X}]_k \to \mathbb{R} \text{ is linear,} \\ & L(1) = 1 \text{ and} \\ & L(M_k) \subseteq \mathbb{R}_{\geq 0} \end{aligned}$$

Theorem. Suppose that L_k solves (P_k) nearly to optimality $(k \in \mathcal{N})$.

$$\forall e \in \mathbb{N} : \forall \varepsilon > 0 : \exists k_0 \in \mathcal{N} \cap [e, \infty) : \forall k \geq k_0 : \exists \mu \in \mathcal{M}^1(S^*) :$$
$$\left\| \left(L_k(X^\alpha) - \int X^\alpha d\mu \right)_{|\alpha| \leq e} \right\| < \varepsilon.$$

Optimization of polynomials on compact semialgebraic sets
SIAM Journal on Optimization **15**, No. 3, 805–825 (2005)

$$(P_k) \quad \text{minimize} \quad L(f) \quad \text{subject to} \quad \begin{aligned} &L : \mathbb{R}[\bar{X}]_k \to \mathbb{R} \text{ is linear,} \\ &L(1) = 1 \text{ and} \\ &L(M_k) \subseteq \mathbb{R}_{\geq 0} \end{aligned}$$

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In particular, if $S^* = \{x^*\}$ is a singleton,

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In particular, if $S^* = \{x^*\}$ is a singleton, then

$$\lim_{k \to \infty} (L_k(X_1), \ldots, L_k(X_n)) = x^*.$$

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Optimization of polynomials on compact semialgebraic sets
SIAM Journal on Optimization **15**, No. 3, 805–825 (2005)
Murray Marshall: Optimization of polynomial functions
Canad. Math. Bull. **46**, 575–587 (2003)
Jean Lasserre: Global optimization with polynomials and the problem of moments
SIAM J. Optim. **11**, No. 3, 796–817 (2001)

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- For fixed k, b depends polynomially on the bitsize of $\mathcal{D}$.
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- b does not depend polynomially on $(\mathcal{D}, k)$.

Further properties of the method

- Feasible solutions of the semidefinite program corresponding to (D_k) give rise to a lower bound a of f^* together with a certificate (advantage) in form of a representation of $f - a$ proving $f - a \in M_k$.

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Detecting optimality and extracting solutions

- If L is an optimal solution of (P_k),
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- If L is an optimal solution of (P_k) which comes from a measure μ on S (criteria of Curto and Fialkow for the truncated S–moment problem), then $L(f) = P_k^* \leq f^* \leq \int f d\mu = L(f)$

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Curto & Fialkow: The truncated complex K–moment problem
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How to solve the relaxations?

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- When going downhill, we could hit the boundary. Therefore we need to be able to compute effectively a so called barrier.
- The cone $S\mathbb{R}_+^{s \times s}$ of positive semidefinite symmetric matrices has such a barrier function:

$$X \mapsto -\log \det X$$

Sums of squares and semidefinite matrices

Let v be a column vector of length s whose entries generate the vector space $\mathbb{R}[\bar{X}]_d$. Then $\sum \mathbb{R}[\bar{X}]_d^2 = \left\{ v^T G v \mid G \in S\mathbb{R}_+^{s \times s} \right\}$.

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$$\sum_{i=1}^t p_i^2 = (Av)^T Av = v^T(\underbrace{A^T A}_{\in S\mathbb{R}_+^{s \times s}})v.$$

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Proof. "$\supseteq$" If $G \in S\mathbb{R}_+^{s \times s}$, then $G = \sum_{i=1}^s x_i x_i^T$ some column vectors $x_1, \ldots, x_s \in \mathbb{R}^s$. Hence $v^T G v = \sum_{i=1}^s (v^T x_i)(x_i^T v) = \sum_{i=1}^s (x_i^T v)^2$.

Shows also that every sum of squares of degree $\leq 2d$ is a sum of s squares.

Translation into a semidefinite program

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$$\mathbb{R}[\bar{X}]_{d_i} \times \mathbb{R}[\bar{X}]_{d_i} \to \mathbb{R} : (p, q) \mapsto L(pqg_i).$$

The semidefinite programs (P_k) and (D_k) one gets in this way are dual to each other.

Pure states on vector spaces

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$$S(E, K, u) = \overline{\operatorname{conv}(\partial_e S(E, K, u))}$$

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where the elements of $\partial_e S(E, K, u)$ are called pure states. A state $\varphi \in S(E, K, u)$ is pure if for all $\varphi_1, \varphi_2 \in S(E, K, u)$,

$$\varphi = \frac{\varphi_1 + \varphi_2}{2} \implies \varphi = \varphi_1 = \varphi_2.$$

Pure states

Theorem. Let E be a real vector space, $K \subseteq E$ be a convex cone with order unit u. Then for every $f \in E$,

$$\varphi(f) > 0 \text{ for all } \varphi \in \partial_e S(E, K, u) \implies f \in K.$$

Definition. Let A be a commutative ring. A subset $M \subseteq A$ is called quadratic module of A if $1 \in M$, $M + M \subseteq M$ and $A^2 M \subseteq M$.

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Theorem (yet unpublished).
If M is an archimedean quadratic module of A, then

$$\partial_e S(A, M, 1) = \{\varphi \mid \varphi : A \to \mathbb{R} \text{ ring homomorphism}, \varphi(M) \subseteq \mathbb{R}_{\geq 0}\}.$$

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Corollary (Jacobi, see the book of Prestel & Delzell).
Let M be an archimedean quadratic module of A. Suppose $f \in A$ such that $\varphi(f) > 0$ for all ring homomorphisms $\varphi : A \to \mathbb{R}$ with $\varphi(M) \subseteq \mathbb{R}_{\geq 0}$. Then $f \in M$.

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Example. Is it true that for $f \in \mathbb{R}[X]$,

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Problem: M is archimedean but if we take $(E, K, u) := (\mathbb{R}[X, Y], M, 1)$, we get $\partial_e S(E, K, u) = [0, 1]$.

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Solution: If we take $(E, K, u) := ((X^k), M \cap (X^k), X^k)$, then

$$\partial_e S(E, K, u) = \left\{ p \mapsto \frac{d^k p}{dX^k}(0) \right\} \cup (0, 1].$$

Pure states on ideals

Theorem (joint work with Sabine Burgdorf). Suppose $M := \sum_{i=0}^{m} \mathbb{R}[\bar{X}]^2 g_i$ is archimedean where $g_i \in \mathbb{R}[\bar{X}]$ and $g_0 := 1$. Set $S := \{g_i \geq 0\}$ and suppose $f \in \mathbb{R}[\bar{X}]$ such that

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Proof. Note that $f \in I$. $\square$

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Claus Scheiderer: Distinguished representations of non-negative...
`http://www.uni-duisburg.de/FB11/FGS/F1/claus.html`

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Tim Netzer: High degree perturbation of nonnegative polynomials, Diplomarbeit Universität Konstanz

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Method is good when f attains a minimum in $\mathbb{R}^n$ since then

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Otherwise, the second implication might badly fail:
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Nie, Demmel, Sturmfels: Minimizing Polynomials via Sum of Squares over the Gradient Ideal
`http://front.math.ucdavis.edu/math.OC/0411342`

New ideas III. Gradient tentacles

Definition. For $f \in \mathbb{R}[\bar{X}]$, define its N-th gradient tentacle for $N \in \mathbb{N}_{\geq 1}$ by $\{x \in \mathbb{R}^n \mid \|\nabla f(x)\| \|x\|^{1+\frac{1}{N}} \leq 1\}$.

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First Theorem (manuscript in preparation).

If $f > 0$ on its N-th gradient tentacle, then

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The proof relies on two non-trivial ingredients:

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- Therefore, my generalization of Schmüdgen's Theorem yields: If $f > 0$ on its N-th gradient tentacle, then

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If f is bounded from below and $f \geq 0$ on all its gradient tentacles, then $f \geq 0$ on $\mathbb{R}^n$.

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The analogue of the Second Theorem for the gradient variety is false:

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The analogue of the Second Theorem for the gradient variety is false: The gradient variety of $(1 - XY)^2 + X^2 - 1$ is $\{0\}$.

New ideas III. Gradient tentacles

Definition. For $f \in \mathbb{R}[\bar{X}]$, define its N-th gradient tentacle for $N \in \mathbb{N}_{\geq 1}$ by $\{x \in \mathbb{R}^n \mid \|\nabla f(x)\| \|x\|^{1+\frac{1}{N}} \leq 1\}$.

First Theorem (manuscript in preparation).
If $f > 0$ on its N-th gradient tentacle, then

$$f \in \sum \mathbb{R}[\bar{X}]^2 + \sum \mathbb{R}[\bar{X}]^2 (1 - \|\nabla f\|^{2N} \|X\|^{2(N+1)}).$$

Second Theorem (follows from Kurdyka, Orro & Simon).
If f is bounded from below and $f \geq 0$ on all its gradient tentacles, then $f \geq 0$ on $\mathbb{R}^n$.

The analogue of the Second Theorem for the gradient variety is false: The gradient variety of $(1 - XY)^2 + X^2 - 1$ is $\{0\}$.

On the other hand, we have countably many tentacles instead of just one gradient variety.