The sums of squares dual of a semidefinite program

(joint work with Igor Klep)

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September 28 - October 1, 2010

Semidefinite programming duality

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(D) & \text{maximize} & -\,tr(A_0 S) \\
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Weak duality: If x is feasible in (P) and (S, a) is feasible in (D), then $\ell(x) \geq a$.

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Weak duality: If x is feasible in (P) and (S, a) is feasible in (D), then $\ell(x) \geq a$. Indeed, $\ell(x) - a = \operatorname{tr}(L(x)S) \geq 0$ since the trace of the product of two positive semidefinite matrices is nonnegative.

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Strong duality: Denote by $P^*, D^* \in \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ the optimal values of (P) and (D) respectively.

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Strong duality: Denote by $P^*, D^* \in \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ the optimal values of (P) and (D) respectively. Suppose that the feasible set of (P) has non-empty interior. Then $P^* = D^*$ (zero gap).

Semidefinite programming duality

A semidefinite program (P) and its standard dual (D) is given by a pencil $L \in \mathbb{R}[\underline{X}]^{m \times m}$ and a linear polynomial $\ell \in \mathbb{R}[\underline{X}]$ as follows:

(P)	minimize	$\ell(x)$	(D)	maximize	a
	subject to	$x \in \mathbb{R}^n$		subject to	$S \in S\mathbb{R}^{m \times m}, a \in \mathbb{R}$
		$L(x) \succeq 0$			$S \succeq 0$
					$\ell - a = \operatorname{tr}(LS)$

Weak duality: If x is feasible in (P) and (S, a) is feasible in (D), then $\ell(x) \geq a$. Indeed, $\ell(x) - a = \operatorname{tr}(L(x)S) \geq 0$ since the trace of the product of two positive semidefinite matrices is nonnegative.

Strong duality: Denote by $P^*, D^* \in \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ the optimal values of (P) and (D) respectively. Suppose that the feasible set of (P) has non-empty interior. Then $P^* = D^*$ (zero gap). Moreover, if $P^* = D^* \in \mathbb{R}$, then (D) attains the common optimal value (dual attainment).

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"$\Longrightarrow$" is strong duality: It is a theorem about existence of a nonnegativity certificate which we prove now for convenience of the auditor.

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In particular, $C_L = \operatorname{im} \varphi$ is closed. □

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$$\ell \geq 0 \text{ on } S_L \iff \ell \in C_L.$$

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Diagonal pencils are never weakly infeasible. For them, Sturm's proposition collapses to Farkas' lemma from linear programming. We want a version of Farkas' lemma characterizing all infeasible pencils. More generally, we want a duality theory for semidefinite programming where strong duality always holds.

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A naive sos Farkas' lemma for semidefinite programming

Observation: If $L \in \mathbb{R}[\underline{X}]^{m \times m}$ is a pencil and $S \in \mathbb{R}[\underline{X}]^{m \times m}$ is an sos-matrix, then $\operatorname{tr}(LS)$ is obviously a polynomial nonnegative on S_L.

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Problems: This gives a way of expressing infeasibility of an SDP by feasibility of another SDP whose size is however exponential. Moreover this is not yet strong duality.

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Idea: By Prestel's theory of semiorderings on a commutative ring, $-\ell^2 \in M_L$ implies that ℓ lies in the real radical

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of the ideal $\operatorname{supp} M_L := M_L \cap -M_L$. If we could get hand on the real radical of this ideal by means of SDP, then we could perhaps "reduce the dimension of the ambient space".

Getting hand on the real radical

For each $d \in \mathbb{N}_0$, let $m_d := \binom{d+n}{n}$ denote the number of monomials of degree at most d in n variables and $\overrightarrow{x_d} \in \mathbb{R}[\underline{X}]^m$ the column vector

$$\overrightarrow{x_d} := \begin{bmatrix} 1 & X_1 & X_2 & \dots & X_n & X_1^2 & X_1X_2 & \dots & \dots & X_n^d \end{bmatrix}^*$$

consisting of these monomials ordered first with respect to the degree and then lexicographic.

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Proposition: Let $d, e \in \mathbb{N}_0$, $m := m_d$ and $k := m_e$.
Let I be a real radical ideal of $\mathbb{R}[\underline{X}]$ and $U \in S\mathbb{R}^{m \times m}$ such that

$$\overrightarrow{x_d}^* U \overrightarrow{x_d} \in I.$$

Suppose $W \in \mathbb{R}^{k \times m}$ with $U \succeq W^* W$,

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$$\overrightarrow{x_e}^* W \overrightarrow{x_d} \in I.$$

Getting hand on the real radical

The following lemma is weak converse.

Lemma: Set $m := m_1$ and $k := m_2$. Suppose $\ell_1, \ldots, \ell_t \in \mathbb{R}[\underline{X}]$ be linear and $q_1, \ldots, q_t \in \mathbb{R}[\underline{X}]$ be quadratic. Let $U \in S\mathbb{R}^{m \times m}$ be such that

$$\overrightarrow{x_1}^* U \overrightarrow{x_1} = \ell_1^2 + \cdots + \ell_t^2.$$

Getting hand on the real radical

The following lemma is weak converse.

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Then there exists $\lambda > 0$ and $W \in \mathbb{R}^{k \times m}$ such that $\lambda U \succeq W^* W$ and

$$\overrightarrow{x_2}^* W \overrightarrow{x_1} = \ell_1 q_1 + \cdots + \ell_t q_t.$$

The sums of squares dual of a semidefinite program

It is now clear that the following provides a duality theory for semidefinite programming where strong duality (zero gap & dual attainment) always holds. Note that the size of the dual (which we do not explicit) is polynomial in the size of the primal.

Theorem: Set $m := m_1$ and $k := m_2$. Let $L \in \mathbb{R}[\underline{X}]^{m\times m}$ be a pencil and $\ell \in \mathbb{R}[\underline{X}]$ be linear. Then $\ell \geq 0$ on S_L if and only if there exist

- quadratic sos-matrices $S_1, \dots, S_n \in \mathbb{R}[\underline{X}]^{m\times m}$,
- matrices $U_1, \dots, U_n \in S\mathbb{R}^{m\times m}$, $W_1, \dots, W_n \in \mathbb{R}^{k\times m}$, $S \in S\mathbb{R}^{m\times m}_{\succeq 0}$ and
- a real number $a \geq 0$

such that

$$\begin{aligned}
&\overrightarrow{x_1}^* U_i \overrightarrow{x_1} + \overrightarrow{x_2}^* W_{i-1} \overrightarrow{x_1} + \operatorname{tr}(LS_i) = 0 && (i \in \{1, \dots, n\}),\\
&U_i \succeq W_i^* W_i && (i \in \{1, \dots, n\}),\\
&\ell + \overrightarrow{x_2}^* W_n \overrightarrow{x_1} = a + \operatorname{tr}(LS)
\end{aligned}$$

where $W_0 := 0$.

Based on other ideas, such a duality theory has also been given by Matt Ramana:

M. Ramana: An exact duality theory for semidefinite programming and its complexity implications
Math. Programming **77** (1997), no. 2, Ser. B, 129–162
http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.47.8540&rep=rep1&type=pdf
http://dx.doi.org/10.1007/BF02614433

See also:

Ramana & Tunçel & Wolkowicz: Strong duality for semidefinite programming
SIAM J. Optim. **7** (1997), Issue 3, 641–662 (1997)
http://www.math.uwaterloo.ca/~ltuncel/publications/strong-duality.pdf
http://dx.doi.org/10.1137/S1052623495288350