SPECTRAHEDRAL RELAXATIONS OF HYPERBOLICITY CONES

MARKUS SCHWEIGHOFER

ABSTRACT. Let p be a real zero polynomial in n variables. Then p defines a rigidly convex set $C(p)$. We construct a linear matrix inequality of size $n+1$ in the same n variables that depends only on the cubic part of p and defines a spectrahedron $S(p)$ containing $C(p)$. The proof of the containment uses the characterization of real zero polynomials in two variables by Helton and Vinnikov. We exhibit many cases where $C(p) = S(p)$.

In terms of optimization theory, we introduce a small semidefinite relaxation of a potentially huge hyperbolic program. If the hyperbolic program is a linear program, we introduce even a finitely convergent hierachy of semidefinite relaxations. With some extra work, we discuss the homogeneous setup where real zero polynomials correspond to homogeneous polynomials and rigidly convex sets correspond to hyperbolicity cones.

The main aim of our construction is to attack the generalized Lax conjecture saying that $C(p)$ is always a spectrahedron. To this end, we conjecture that real zero polynomials in fixed degree can be "amalgamated" and show it in three special cases with three completely different proofs. We show that this conjecture would imply the following partial result towards the generalized Lax conjecture: Given finitely many planes in $\mathbb{R}^n$, there is a spectrahedron containing $C(p)$ that coincides with $C(p)$ on each of these planes. This uses again the result of Helton and Vinnikov.

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1. INSTEAD OF AN INTRODUCTION

I started to write these notes during my sabbatical winter term 2018/2019 but still did not finish them due to lack of time. The notes are thus still incomplete and probably contain numerous errors. I do not yet know what is the best way to publish them. All readers are kindly invited to report any errors (from typographic to fatal), e.g., by electronic mail to:

`markus.schweighofer@uni-konstanz.de`

Suggestions and remarks are also highly welcome. I will try to correct and expand these notes in the future. This version is still far from being suitable for publication and I will not yet submit it to a journal. The structure of the notes or even the title will change in the future. If you want to reference some content in the future, please check for newer or even published versions of the text. I nevertheless decided to make the notes available on the `arXiv` preprint server since I plan to give some talks about the material.

This article lacks a proper introduction. We consider rigidly convex sets in the affine setting and hyperbolicity cones in the homogeneous setting [Vin2]. While we will formally introduce these notions, the reader who is not yet familiar with them should first have a look at some survey articles [Wag, Pem, Fis]. We will assume that the reader knows about the definition and the most basic properties of spectrahedra [Lau].

An extremely important theorem, the Helton-Vinnikov theorem [HV, Vin1, Han] and some weaker versions or predecessors of it [GKVW, PV] (see [HV, §8] for the complicated history) show that in dimension two each rigidly convex set is a spectrahedron. In the homogeneous setting, it shows that in dimension three each hyperbolicity cone is spectrahedral. To my knowledge, all existing proofs of this theorem are very deep [HV, Vin1, Han] and not easily accessible. Although it complicates a bit our approach, we therefore take care to use whenever possible only a weaker version of the theorem (with hermitian instead of symmetric matrices) whose proof is considerably easier. Perhaps the most elementary proof of it can be found in [GKVW].

The open question whether, regardless of the dimension, every rigidly convex set is a spectrahedron, or in the homogeneous setting whether each hyperbolicity cone is spectrahedral, is now widely known as the generalized Lax conjecture (GLC). This conjecture that has motivated our work has first been formulated in [HV, Subsection 6.1] together with a stronger conjecture which turned out to be false [Vin2, Proposition 7] as Brändén showed [**?**].

There are a number of very interesting partial results towards the Generalized Lax Conjecture, see [HV, BK, **?**, Ami, Kum2, Sau] and the references therein. Some of the constructions in these references lead to huge linear matrix inequalities describing hyperbolicity cones [**?**, Ami]. In the light of the recent result [RRSW], the huge size of the matrices is no longer surprising although [RRSW] does says little for these concrete examples.

Our basic idea here is that we produce very small size natural spectrahedral outer approximations to rigidly convex sets and hyperbolicity cones. To the best of our knowledge, these are new although they turn out to be more or less known for quadratic real zero and hyperbolic polynomials and they are related to a construction of [Sau] in the case of the “general” spectrahedron. The fact that our constructions yield actually a relaxation seems to be non-trivial and relies on the Helton-Vinnikov theorem or some weaker versions of it. In view of the lower complexity bounds proved in [RRSW] for the size of a describing linear matrix inequality, our approach seems at first sight hopeless since we produce linear matrix inequalities of very small size.

However, there is an important additional twist which allows us to prove a partial result and leaves room for speculations whether GLC could be attacked in the same way. The basic idea is to try to apply our construction to a new real zero polynomial in the original variables and many more new variables. The new real zero polynomial should yield back the original one when all new variables are set to zero. Due to the big number of new variables, the linear matrix inequality we now produce has huge size in perfect accordance with the results of [RRSW].

We present a conjecture that would allow to "amalgamate" two real zero polynomials that agree whenever you set all variables to zero that do not appear in both polynomials. By "amalgamating" we mean to find a third real zero polynomial that yields the first or second polynomial, respectively, when you set all the non-corresponding variables to zero. We conjecture that this amalgamation is always possible. For our purpose, it would not be a problem if the degree of the amalgamating polynomial exceeded the degree of both polynomials. However, we conjecture that such a degree rise is not necessary. We show that our amalgamation conjecture holds in three cases, namely if the two polynomials do not share any variables, if each of them is a univariate polynomial and if each of them is quadratic. We handle each of these three special cases in a completely different way, namely by the theory of stability preservers, the Helton-Vinnikov theorem and the theory of positive-semidefinite matrix completion, respectively.

Our main result is that our amalgamation conjecture (but unfortunately none of its three special cases we will prove) would have implications on GLC. We show that it would imply that each rigidly convex set can be "wrapped" into a spectrahedron. By "wrapping" we mean that the rigidly convex set can be packed into a spectrahedron which is tied to the rigidly convex set with finitely many cords. A bit less vaguely, the cords are one-dimensional curves lying on the boundary of both, the rigidly convex set and the spectrahedron. Mathematically speaking, for each rigidly convex set and finitely given many given planes (two-dimensional subspaces) there would always exist a spectrahedron containing the rigidly convex set and coinciding with it on the given planes. In proving this, we use once more the Helton-Vinnikov theorem or a weaker version of it.

The prototype of all rigidly convex sets are polyhedra. Curiously, our work can even be applied to yield a spectrahedral relaxation of a polyhedron. This makes sense if the polyhedron has a large number of facets since we produce a small size linear matrix inequality. In this case, we can even make a hierachy of relaxations out of our construction which converges finitely. Just like in the Lasserre hierarchy of semidefinite relaxations for polynomial optimization problems [Lau], moment and localization matrices play a big role. Here, we use moment and localization matrices filled with real numbers to define our relaxation. In the Lasserre hierarchy, the matrices are filled with unknowns which one hopes to become moments when solving the semidefinite program. Moment matrices play thus a completely different role here than in the Lasserre hierarchy.

What seems to be strange at first sight is that in general (in the non-LP-case) our relaxations are based on "moments" (actually a kind of pseudo-moments which is different from the notion of pseudo-moments in the Lasserre context) up to degree three only. This makes however a lot of sense for people that are acquainted with tensor decomposition methods like Jennrich's algorithm [Har] many of which are also based on degree three moments.

Our "pseudo-moments" correspond to real zero polynomials. When the real zero polynomial is a product of linear polynomials, these are actual moments of a sum of Dirac measures. When the real zero polynomial has more generally a certain symmetric determinantal representation, we deal with certain "tracial moments".

Some of our results, like Proposition 3.8 or Proposition 3.14 are related to the question if there is a generalization of certain operations from moments to pseudo-moments. Namely, one can rotate and shift a point-configuration and the behavior of the moments of the corresponding sum of Dirac measures is mimicked. This will be another line of future research whose applications are yet unclear.

Let us say a word why we work mainly with rigidly convex sets before going to hyperbolicity cones (i.e., we prefer, at least in a first stage, to work in the affine setting rather than in the homogeneous one): Comparing Definitions 3.19 and 6.18, it becomes clear that in the homogeneous setup, there are more technical steps in our construction which makes also all the proofs a bit more confusing. Moreover, GLC can be perfectly formulated in the setting of rigidly convex sets. Also, rigidly convex sets of very small dimension can more easily be visualized. In general, we feel the more natural setup for our method is the affine one although the homogeneous one is also important and cannot be simply deduced from the affine one.

1.1. **Numbers.** We always write $\mathbb{N}$ and $\mathbb{N}_0$ for the sets of positive and nonnegative integers, $\mathbb{R}$ and $\mathbb{C}$ for the fields of real and complex numbers, respectively. We denote the complex imaginary unit by $\mathbb{i}$ so that the letter i can be used for other purposes such as summations.

1.2. **Polynomials and power series.** Let K be a field. We will always denote by $x = (x_1, \ldots, x_n)$ an n-tuple of distinct variables so that

$$K[x] := K[x_1, \ldots, x_n] \qquad \text{and} \qquad K[[x]] := K[[x_1, \ldots, x_n]]$$

denote the rings of polynomials and (formal) power series in n variables over K, respectively. The number n of variables will thus often be fixed implicitly although suppressed from the notation. Sometimes, we need an additional variable x_0 so that $K[x_0, x] = K[x_0, \ldots, x_n]$ denotes the ring of polynomials in $n+1$ variables over K. We allow of course the case $n = 0$ since it is helpful to avoid case distinctions, for example in inductive proofs. We also use the letter t to denote another variable so that $K[t]$ and $K[[t]]$ denote the rings of univariate polynomials and formal power series over K, respectively. For $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$, we denote

$$|\alpha| := \alpha_1 + \ldots + \alpha_n$$

and call

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

a monomial. Hence

$$K[[x]] = \left\{ \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha x^\alpha \mid a_\alpha \in K \right\}.$$

The degree $\deg p$ of a power series $p = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha x^\alpha$ with all $a_\alpha \in K$ is defined as a supremum in the (naturally) ordered set $\{-\infty\} \cup \mathbb{N}_0 \cup \{\infty\}$ by

$$\deg p := \sup\{|\alpha| \mid a_\alpha \neq 0\} \in \{-\infty\} \cup \mathbb{N}_0 \cup \{\infty\}$$

which entails $\deg 0 = -\infty$ and

$$K[x] = \{p \in K[[x]] \mid \deg p < \infty\}.$$

We call a polynomial constant, linear, quadratic, cubic and so on if its degree is *less than or equal to* 0, 1, 2, 3 and so on.

1.3. **Matrices.** For any set S, we denote by $S^{m\times n}$ the set of all matrices with m columns and n rows over S. It is helpful to allow the *empty matrix* which is for each $m, n \in \mathbb{N}_0$ the only element of $S^{m\times 0} = S^{0\times n}$. For any matrix $A \in S^{m\times n}$, we denote by $S^T \in S^{n\times m}$ its transpose. For a matrix $A \in \mathbb{C}^{m\times n}$, we denote by $A^* \in \mathbb{C}^{n\times m}$ its *complex conjugate transpose*.

We tend to view n-tuples over a set S, i.e., elements of S^n, as *column vectors*, i.e., elements of $S^{n\times 1}$. In this sense, we can transpose column vectors and get *row vectors* or vice versa. Analogously, we can apply the complex conjugate transpose to complex column vectors to get row vectors and vice versa.

If R is a commutative ring (where a ring in our sense is always associative with 1), $d \in \mathbb{N}_0$ and $A \in R^{d\times d}$, then the determinant $\det A$ is declared in the usual way. For each $n \in \mathbb{N}_0$, we denote by $I_n \in R^{d\times d}$ the unity matrix of size d for which $\det I_d = 1$ (even when $d = 0$, i.e., the determinant of the empty matrix is one).

A matrix $A \in \mathbb{C}^{d\times d}$ is called *hermitian* if $A = A^*$ which is equivalent to $v^*Av \in \mathbb{R}$ for all $v \in \mathbb{C}^d$. A matrix $U \in \mathbb{C}^{d\times d}$ is called *unitary* if $U^*U = I_d$ which is equivalent to $UU^* = I_d$ and also to $\|Uv\| = \|v\|$ for all $v \in \mathbb{C}^d$. We will use many times and often without mentioning the spectral theorem for hermitian matrices that says that for any hermitian $A \in \mathbb{C}^{d\times d}$ there exists a unitary $U \in \mathbb{C}^{d\times d}$ such that U^*AU is a diagonal matrix. In this situation, the diagonal matrix is obviously real and the diagonal elements are the eigenvalues of A.

A hermitian matrix all of whose (real) eigenvalues are nonnegative is called *positive semidefinite (psd)*. If all its eigenvalues are even positive, it is called *positive definite (pd)*. If all its eigenvalues are nonpositive or negative then it is called *negative semidefinite (nsd)* or *negative definite (nd)*, respectively. A *definite matrix* is one that is pd or nd. It is easy to see that a matrix $A \in \mathbb{C}^{d\times d}$ is psd if and only if $v^*Av \geq 0$ (i.e., v^*Av is real and nonnegative) for all $v \in \mathbb{C}^n$. The analogous characterizations for pd, nsd and nd matrices should be clear. Using the intermediate value theorem, it follows that $A \in \mathbb{C}^{d\times d}$ is definite if and only if $v^*Av \in \mathbb{R} \setminus \{0\}$ for all $v \in \mathbb{C}^d \setminus \{0\}$.

For matrices $A, B \in \mathbb{C}^{d\times d}$, we write $A \preceq B$ (or equivalently $B \succeq A$) if $B - A$ is psd and $A \prec B$ (or equivalently $B \succ A$) if $B - A$ is pd.

A real matrix $A \in \mathbb{R}^{d\times d}$ is of course symmetric if and only if it is hermitian. It is called *skew-symmetric* if $A = -A^T$. A real unitary matrix is called *orthogonal*. A matrix $U \in \mathbb{R}^{d\times d}$ is thus obviously orthogonal if $\|Uv\| = \|v\|$ for all $v \in \mathbb{R}^d$. The spectral theorem for symmetric matrices that we will often use silently says that for any symmetric $A \in \mathbb{R}^{d\times d}$ there exists an orthogonal $U \in \mathbb{R}^{d\times d}$ such that U^*AU is a diagonal matrix. A real matrix is obviously psd if and only if it is symmetric and its (real) eigenvalues are nonnegative. It is thus easy to see that a matrix $A \in \mathbb{R}^{d\times d}$ is psd if and only if A is symmetric and $v^*Av \geq 0$ for all $v \in \mathbb{R}^n$. The analogous characterizations for pd, nsd and nd matrices are clear. Moreover, $A \in \mathbb{R}^{d\times d}$ is definite if and only if A is symmetric and $v^*Av \neq 0$ for all $v \in \mathbb{R}^d \setminus \{0\}$. In particular, the notation $A \succeq 0$ means for $A \in \mathbb{R}^{d\times d}$ that A is symmetric and $v^*Av \geq 0$ for all $v \in \mathbb{R}^d$.

If $A, B \in \mathbb{R}^{d\times d}$, then $C := A + \mathbb{i}B \in \mathbb{C}^{d\times d}$ is hermitian if and only if A is symmetric and B is skew-symmetric, or in other words, if the matrix

$$R := \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathbb{R}^{(2d)\times(2d)}$$

is symmetric. Now if C is hermitian, one easily checks that

$$(v + \mathbb{i}w)^* C(v + \mathbb{i}w) = \begin{pmatrix} v \\ w \end{pmatrix}^T R \begin{pmatrix} v \\ w \end{pmatrix}$$

for all $v, w \in \mathbb{R}^n$. Hence it is clear that C is psd if and only if R is psd.

1.4. **Matrix polynomials.** Matrices whose entries are polynomials are often called *matrix polynomials*. As said above, we allow the empty matrix and therefore the empty matrix polynomial. We define the *degree* of the empty matrix polynomial to be ∞ and the degree of a non-empty matrix polynomial to be the maximum of the degrees of its entries. Hence the degree of a matrix polynomial is always from $\mathbb{N}_0$ except for the zero matrix polynomial which has degree ∞. Exactly as for polynomial, we say that a matrix polynomial is constant, linear, quadratic, cubic and so on if its degree is *less than or equal to* 0, 1, 2, 3 and so on. In numerical linear algebra, linear matrix polynomials are often called matrix pencils, especially if they are univariate, i.e., only one variables is involved. Here it will be convenient to reserve the term *pencil* for *symmetric* linear real matrix polynomials in one or several variables. A pencil of size d in n variables $x = (x_1, \dots, x_n)$ is thus of the form

$$A_0 + x_1 A_1 + \ldots + x_n A_n$$

where $A_0, A_1, \dots, A_n \in \mathbb{R}^{d\times d}$ are symmetric. The determinant of such a pencil is of course a polynomial of degree at most d.

1.5. **Convex sets and cones.** A subset C of a real vector space V is called *convex* if it contains with any two of its points $v, w \in C$ also the line segment

$$\{\lambda v + (1 - \lambda)w \mid \lambda \in [0, 1]\}$$

joining them. We call it a *cone* what many authors call a "convex cone", namely a subset C of a real vector space V that contains the origin, is closed under addition and under multiplication with nonnegative scalars, i.e., $0 \in C$, $v + w \in C$ for all $v, w \in C$ and $\lambda v \in C$ for all $v \in C$ and $\lambda \geq 0$. For example, the set

$$\{A \in \mathbb{R}^{d\times d} \mid A \succeq 0\}$$

of psd matrices is a cone inside the vector space $\mathbb{R}^{d\times d}$ of matrices of size d. Most of the convex sets and cones we will consider will however live in the vector space $\mathbb{R}^d$.

1.6. **Affine half spaces, polyhedra and spectrahedra.** A subset of $\mathbb{R}^n$ of the form $\{a \in \mathbb{R}^n \mid \ell(a) \geq 0\}$ where $\ell \in \mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$ is a non-constant linear polynomial is called an *affine half space*. A finite intersection of such half spaces is called a *polyhedron*. A subset of $\mathbb{R}^n$ of the form $\{a \in \mathbb{R}^n \mid L(a) \succeq 0\}$ where $L \in \mathbb{R}[x]^{d\times d}$ is a pencil of size d for some $d \in \mathbb{N}_0$ is called a *spectrahedron*. Hence polyhedra are exactly the spectrahedra that can be defined by diagonal pencils. Spectrahedra are more flexible than polyhedra and unlike polyhedra allow for round shapes in their geometry. On the other hand, they still share many good properties with polyhedra. Restating the definition more explicitly, $S \subseteq \mathbb{R}^n$ is a spectrahedron if and only if there exists $d \in \mathbb{N}_0$ and symmetric matrices $A_0, A_1, \dots, A_n \in \mathbb{R}^{d\times d}$ such that

$$S = \{a \in \mathbb{R}^n \mid A_0 + a_1A_1 + \ldots + a_nA_n \succeq 0\}.$$

Here one could equivalently require $A_0, A_1, \dots, A_n$ to be complex hermitian matrices instead of symmetric real matrices as can be seen easily by the above translation of a psd condition for a complex matrix into a psd condition of a real matrix of double size. Using block diagonal matrices, one sees immediately that finite intersections of spectrahedra are again spectrahedra.

1.7. **Elementary combinatorics.** We use some standard notation from elementary combinatorics. For $\ell \in \mathbb{N}_0$, the factorial of ℓ

$$\ell! := \ell(\ell-1)\cdots 1$$

denotes the number of permutations of ℓ objects (in particular $0! = 1$). For $k, \ell \in \mathbb{N}_0$, the binomial coefficient

$$\binom{\ell}{k} := \frac{\ell!}{(\ell-k)!k!}$$

denotes the number of choices of k objects among ℓ. For $\alpha \in \mathbb{N}_0^n$, the multinomial coefficient

$$\binom{|\alpha|}{\alpha} := \binom{\alpha_1 + \ldots + \alpha_n}{\alpha_1 \ldots \alpha_n} := \frac{|\alpha|!}{\alpha_1! \ldots \alpha_n!}$$

denotes the number of ways of depositing $|\alpha|$ distinct objects into n distinct bins, with α_i objects in the i-th bin for each $i \in \{1, \dots, n\}$. For $n = 1$ this notation agrees with the one for binomial coefficients which fortunately does not lead to a conflict.

2. Real zero polynomials and spectrahedra

2.1. **Definition and examples.** The following definition stems from [HV, §2.1].

Definition 2.1. We say that $p \in \mathbb{R}[x]$ is a *real zero polynomial* if for all $a \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$,

$$p(\lambda a) = 0 \implies \lambda \in \mathbb{R}.$$

Remark 2.2. If $p \in \mathbb{R}[x]$ is a real zero polynomial, then $p(0) \neq 0$.

Proposition 2.3. Let $p \in \mathbb{R}[x]$. Then p is a real zero polynomial if and only if for each $a \in \mathbb{R}^n$, the univariate polynomial

$$p(ta) = p(ta_1, \dots, ta_n) \in \mathbb{R}[t]$$

splits (i.e., is a product of non-zero linear polynomials) in $\mathbb{R}[t]$.

Proof. The "if" direction is easy and the "only if" direction follows from the fundamental theorem of algebra. □

Example 2.4. Let $p \in \mathbb{R}[x]$ be a quadratic real zero polynomial with $p(0) = 1$. Then p can be uniquely written as

$$p = x^TAx + b^Tx + 1$$

with a symmetric matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$. For $a \in \mathbb{R}^n$ the univariate quadratic polynomial $p(ta) = a^TAat^2 + b^Tat + 1$ splits in $\mathbb{R}[t]$ if and only if its discriminant $(b^T a)^2 - 4a^T Aa = a^T(bb^T - 4A)a$ is is nonnegative. Hence p is a real zero polynomial if and only if

$$bb^T - 4A \succeq 0.$$

Lemma 2.5. Let $A, B \in \mathbb{C}^{d\times d}$ be hermitian and suppose A is definite. Then for all $\lambda \in \mathbb{C}$,

$$\det(A + \lambda B) = 0 \implies \lambda \in \mathbb{R}.$$

Proof. The case $A \prec 0$ can be reduced to the case $A \succ 0$ by scaling p with $(-1)^d$. WLOG $A \succ 0$. Since there is a unitary matrix $U \in \mathbb{C}^{d\times d}$ such that U^*AU is diagonal, we can assume that A is diagonal. The diagonal entries $d_1, \dots, d_n$ of A are positive. Multiplying both A and B from the left and the right by the diagonal matrix whose diagonal entries are the inverted square roots of $d_1, \dots, d_n$ changes the determinant of $A + \lambda B$ but preserves the condition $\det(A + \lambda B) = 0$. Hence WLOG $A = I_d$. Now $\det(B - (-\frac{1}{\lambda})I_d) = 0$. But then $-\frac{1}{\lambda}$ is an eigenvalue of the hermitian matrix B and thus real. Hence λ is real as well. $\square$

The most obvious example of real zero polynomials are products of linear polynomials that do not vanish at the origin. But this is just the special case where all A_i are diagonal matrices of the following more general example:

Proposition 2.6. Let $A_0, A_1, \dots, A_n \in \mathbb{C}^{d\times d}$ be hermitian matrices such that A_0 is definite and

$$p = \det(A_0 + x_1 A_1 + \ldots + x_n A_n),$$

then p is a real zero polynomial.

Proof. If $a \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ with $p(\lambda a) = 0$, then $\det(A_0 + \lambda B) = 0$ where $B := a_1 A_1 + \ldots + a_n A_n \in \mathbb{C}^{d\times d}$ and Lemma 2.5 implies $\lambda \in \mathbb{R}$. $\square$

2.2. **The Helton-Vinnikov theorem.** The following celebrated partial converse to Proposition 2.6 has been obtained in 2006 by Helton and Vinnikov [HV, Theorem 2.2, §4]. For a short account of the long history of partial results, we refer to [HV, §8]. Recently, a purely algebraic proof of this theorem has been given by Hanselka [Han, Theorem 2].

Theorem 2.7 (Helton and Vinnikov)**.** If $p \in \mathbb{R}[x_1, x_2]$ is a real zero polynomial of degree d with $p(0) = 1$, then there exist symmetric $A_1, A_2 \in \mathbb{R}^{d\times d}$ such that

$$p = \det(I_d + x_1 A_1 + x_2 A_2).$$

A weaker version of this theorem will be enough for most of our purposes. Since this weaker version seems to be considerably easier to prove (see mainly [GKVW], also [PV] and [Han, §7]), we state it here. this weaker version as a corollary. In the following, we will always prefer to use the corollary instead of the theorem.

Corollary 2.8 (Helton and Vinnikov)**.** If $p \in \mathbb{R}[x_1, x_2]$ is a real zero polynomial of degree d with $p(0) = 1$, then there exist hermitian matrices $A_1, A_2 \in \mathbb{C}^{d\times d}$ such that

$$p = \det(I_d + x_1 A_1 + x_2 A_2).$$

Example 2.9. Consider a quadratic real zero polynomial $p \in \mathbb{R}[x_1, x_2]$ with $p(0) = 1$. Write

$$p = x^T A x + b^T x + 1$$

with a symmetric matrix $A \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^n$. Write moreover

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Then $bb^T - 4A \succeq 0$ by Example 2.4. Hence the leading prinicipal minors $r := b_1^2 - 4a_{11}$ and $s := \det(bb^T - 4A)$ are nonnegative. Consider the real symmetric matrices

$$A_1 := \frac{1}{2} \det \begin{pmatrix} b_1 - \sqrt{r} & 0 \\ 0 & b_1 + \sqrt{r} \end{pmatrix}$$

and

$$A_2 := \frac{1}{2r} \begin{pmatrix} b_1^2 b_2 - b_1 b_2 \sqrt{r} - 4a_{11} b_2 + 4a_{12}\sqrt{r} & \sqrt{rs} \\ \sqrt{rs} & b_1^2 b_2 + b_1 b_2 \sqrt{r} - 4a_{11} b_2 - 4a_{12}\sqrt{r} \end{pmatrix}.$$

Then one can easily verify that $p = \det(I_2 + x_1 A_1 + x_2 A_2)$.

2.3. **Rigidly convex sets.**

Definition 2.10. Let $p \in \mathbb{R}[x]$ be a polynomial. Then we call

$$Z(p) := \{a \in \mathbb{R}^n \mid p(a) = 0\}$$

the (real) *zero set* defined by p.

Definition 2.11. Let $p \in \mathbb{R}[x]$ be a real zero polynomial. Then we call

$$C(p) := \{a \in \mathbb{R}^n \mid \forall \lambda \in [0,1) : p(\lambda a) \neq 0\}$$

the *rigidly convex set* defined by p.

A priori, it is not even clear that rigidly convex sets are convex. This was however already known to Gårding, see Theorem 2.15 below. In the case where p has a determinantal representation of the kind considered in Proposition 2.6 above, it is however easy to show that $C(p)$ is not only convex but even a spectrahedron:

Proposition 2.12. Suppose $d \in \mathbb{N}_0$, $A_0, A_1, \ldots, A_n \in \mathbb{C}^{d \times d}$ be hermitian, $A_0 \succ 0$ and

$$p = \det(A_0 + x_1 A_1 + \ldots + x_n A_n).$$

Then

$$C(p) = \{a \in \mathbb{R}^n \mid A_0 + a_1 A_1 + \ldots + a_n A_n \succeq 0\}$$

and

$$C(p) \setminus Z(p) = \{a \in \mathbb{R}^n \mid A_0 + a_1 A_1 + \ldots + a_n A_n \succ 0\}.$$

Proof. The second statement follows easily from the first. To prove the first, let $a \in \mathbb{R}^n$ and set $B := a_1 A_1 + \ldots + a_n A_n$. We have to show

$$(\forall \lambda \in [0,1) : \det(A_0 + \lambda B) \neq 0) \iff A_0 + B \succeq 0.$$

Since A_0 is positive definite, there exists a (unique) pd matrix $\sqrt{A_0}$ whose square is A_0. Rewriting both the left and right hand side of our claim, it becomes

$$(\forall \lambda \in [0,1) : \det(I_d + \lambda C) \neq 0) \iff I_d + C \succeq 0.$$

where $C := \sqrt{A_0}^{-1} B \sqrt{A_0}^{-1}$. Since C is hermitian, we find a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that $U^* C U$ is a diagonal matrix with diagonal entries $d_1, \dots, d_n \in \mathbb{R}$. Our claim can be rewritten

$$\left(\forall \lambda \in [0,1) : \prod_{i=1}^{d} (1 + \lambda d_i) \neq 0 \right) \iff \forall i \in \{1, \dots, n\} : 1 + d_i \geq 0$$

which is easily checked. □

Remark 2.13. For $n \in \{0,1\}$, it is trivial that each rigidly convex set in $\mathbb{R}^n$ is a spectrahedron. For $n = 2$ this follows from Helton-Vinnikov Corollary 2.8 together with Proposition 2.12. Whether this continues to holds for $n > 2$ is unknown and is the topic of Section 8 below.

If $S \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}^n$, we write $S + a := \{b + a \mid b \in S\}$. Gårding proved the following result in a more elementary way [Går]. For convenience of the reader, we include here a proof but allow ourselves the luxury to base it on the Helton-Vinnikov Corollary 2.8 although this is an overkill.

Theorem 2.14 (Gårding)**.** Let $p \in \mathbb{R}[x]$ be a real zero polynomial and $a \in C(p) \setminus Z(p)$. Then the shifted polynomial $p(x + a)$ is a real zero polynomial as well and $C(p) = C(p(x + a)) + a$.

Proof. We first show that $p(x + a)$ is a real zero polynomial. To this end, let $b \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ such that $p(\lambda b + a) = 0$. We have to show $\lambda \in \mathbb{R}$. We have $b \neq 0$ since $a \notin Z(p)$. If $a = \mu b$ for some $\mu \in \mathbb{R}$, then $p((\lambda + \mu)a) = 0$ implies $\lambda + \mu \in \mathbb{R}$ and thus $\lambda \in \mathbb{R}$. Hence we can now suppose that a and b are linearly independent. By an affine transformation, we can even suppose that a and b are the first two unit vectors in $\mathbb{R}^n$. Without loss of generality, we can thus assume that the number of variables is $n = 2$. Also WLOG $p(0) = 1$. By the Helton-Vinnikov Corollary 2.8, we can write

$$p = \det(I_d + x_1 A_1 + x_2 A_2)$$

with hermitian $A_1, A_2 \in \mathbb{C}^{d \times d}$ where $d := \deg p$. The hypothesis $a \in C(p) \setminus Z(p)$ now translates into $A := I_d + A_1 \succ 0$ by Proposition 2.12. From $\det(A + \lambda A_2) = 0$ and Lemma 2.5, we get $\lambda \in \mathbb{R}$.

To prove the second statement, let $b \in \mathbb{R}^n$. We show that

$$(*) \qquad b \in C(p) \iff b - a \in C(p(x + a)).$$

The case where a and b are linearly dependent is an easy exercise. Suppose therefore that a and b are linearly independent. After an affine transformation, we can even assume that a and b are the first two unit vectors. Hence we can reduce to the case where the number of variables n equals 2. By the Helton-Vinnikov Corollary, we can choose hermitian matrices $A_1, A_2 \in \mathbb{C}^{d \times d}$ such that

$$p = \det(I_d + x_1 A_1 + x_2 A_2)$$

so that

$$C(p) = \{c \in \mathbb{R}^2 \mid I_d + c_1 A_1 + c_2 A_2 \succeq 0\}$$

by Proposition 2.12. Then

$$p(x + a) = \det((I_d + A_1) + x_1 A_1 + x_2 A_2)$$

and $I_d + A_1 \succ A_1 \succ 0$ so that

$$C(p(x + a)) = \{c \in \mathbb{R}^2 \mid I_d + A_1 + c_1 A_1 + x_2 A_2 \succeq 0\}.$$

again by Proposition 2.12. Our claim $(*)$ now translates into

$$I_d + A_2 \succeq 0 \iff I_d + A_1 - A_1 + A_2 \succeq 0.$$

which holds trivially. □

Theorem 2.15 (Gårding)**.** If p is a real zero polynomial, then $C(p) \setminus Z(p)$ and $C(p)$ are convex.

Proof. Call for the moment a subset $S \subseteq \mathbb{R}^n$ *star-shaped* if for all $x \in S$, we have $\lambda x \in S$ for each $\lambda \in [0,1]$. Clearly, a subset $S \subseteq \mathbb{R}^n$ is convex if and only if $S - a$ is star-shaped for each $a \in S$. To show that $C(p) \setminus Z(p)$ is convex, we therefore fix $a \in C(p) \setminus Z(p)$ and show that $(C(p) \setminus Z(p)) - a$ is star-shaped. By Theorem 2.14, $(C(p) \setminus Z(p)) - a$ equals $C(q) \setminus Z(q)$ for some real zero polynomial $q \in \mathbb{R}[x]$ and therefore is obviously star-shaped by Definition 2.11.

Finally, to prove that $C(p)$ is convex, note that

$$\begin{aligned} C(p) = \{a \in \mathbb{R}^n \mid \forall \lambda \in (0,1) : \lambda a \in \ C(p) \setminus Z(p)\} \\ = \bigcap_{\lambda \in (0,1)} \{a \in \mathbb{R}^n \mid \lambda a \in \ C(p) \setminus Z(p)\} \end{aligned}$$

is an intersection of convex sets and therefore convex. □

3. The relaxation

3.1. **The linear form associated to a polynomial.** All power series are formal. We refer to [God, §3], [Rob, §6.1] and [Rui, §2] for an introduction to power series.

Definition 3.1. Suppose $a_\alpha \in \mathbb{R}$ for all $\alpha \in \mathbb{N}_0^n$ and consider the power series

$$p = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha x^\alpha \in \mathbb{R}[[x]].$$

Then we call for $d \in \mathbb{N}_0$, the polynomial

$$\operatorname{trunc}_d p := \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \le d}} a_\alpha x^\alpha \in \mathbb{R}[x]$$

the *truncation of p at degree d*.

Definition 3.2. Let $p \in \mathbb{R}[[x]]$ be a power series.

(a) If p has constant coefficient 0, then the power series

$$\exp p := \sum_{k=0}^{\infty} \frac{p^k}{k!} \in \mathbb{R}[[x]]$$

is well-defined because all monomials appearing in p^k have degree at least k so that

$$\operatorname{trunc}_d \exp p = \operatorname{trunc}_d \sum_{k=0}^{d} \frac{p^k}{k!}$$

for all $d \in \mathbb{N}_0$. We call it the *exponential* of q.

(b) If p has constant coefficient 1, then the power series

$$\log p := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(p-1)^k \in \mathbb{R}[[x]]$$

is well-defined because all monomials appearing in $(p-1)^k$ have degree at least k so that

$$\operatorname{trunc}_d \log p = \operatorname{trunc}_d \sum_{k=1}^{d} \frac{(-1)^{k+1}}{k}(p-1)^k$$

for all $d \in \mathbb{N}_0$. We call it the *logarithm* of p.

The following can be found for example in [God, Lemma 4.1], [Rob, §5.4.2, Proposition 2] or [Rob, §6.1.3]

Proposition 3.3. Consider the sets

$$A := \{p \in \mathbb{R}[[x]] \mid \operatorname{trunc}_0 p = 0\} \qquad \text{and}$$
$$B := \{p \in \mathbb{R}[[x]] \mid \operatorname{trunc}_0 p = 1\}.$$

Then the following hold:
(a) $\exp\colon A \to B$ and $\log\colon B \to A$ are inverse to each other.
(b) $\exp(p+q) = (\exp p)(\exp q)$ for all $p, q \in A$
(c) $\log(pq) = (\log p) + (\log q)$ for all $p, q \in B$

Proof. (a) can be proven in two different ways: One way is to play it back to known facts about converging power series from calculus [Rob, §5.4.2, Proposition 2]. Note that the argument given in [God, Lemma 4.1] looks innocent but in reality needs good knowledge of multivariate power series [Rui, §I.1, §I.2]. The other way to prove it is by using formal composition and derivation of power series: Deduce the result from the univariate case $n = 1$ by formally deriving the formal composition (in either way) of the univariate logarithmic and exponential power series [Rob, §6.1.3].

(b) is an easy calculation:

$$\exp(p+q) = \sum_{k=0}^{\infty} \frac{(p+q)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} p^i q^{k-i}$$
$$= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{p^i}{i!} \frac{q^{k-i}}{(k-i)!} = (\exp p)(\exp q)$$

(c) follows easily from (a) and (b). □

Definition 3.4. Let $p \in \mathbb{R}[[x]]$ satisfy $p(0) \neq 0$ and let $d \in \mathbb{N}_0$. We define the linear form $L_{p,d}$ on $\mathbb{R}[x]$ *associated to* p *with respect to the virtual degree* d by specifying it on the monomial basis of $\mathbb{R}[x]$, namely by setting

$$L_{p,d}(1) = d$$

and by requiring the identity of formal power series

$$-\log \frac{p(-x)}{p(0)} = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha \neq 0}} \frac{1}{|\alpha|} \binom{|\alpha|}{\alpha} L_p(x^\alpha) x^\alpha$$

to hold. If $p \in \mathbb{R}[x]$, then we call $L_p := L_{p,\deg f}$ the linear form *associated to* p.

Example 3.5. Suppose $p \in \mathbb{R}[[x]]$ such that

$$\operatorname{trunc}_3 p = 1 + \sum_{i \in \{1,\dots,n\}} a_i x_i + \sum_{\substack{i,j \in \{1,\dots,n\} \\ i \leq j}} a_{ij} x_i x_j + \sum_{\substack{i,j,k \in \{1,\dots,n\} \\ i \leq j \leq k}} a_{ijk} x_i x_j x_k$$

where $a_i, a_{ij}, a_{ijk} \in \mathbb{R}$. Then

$$\begin{aligned}\operatorname{trunc}_3(-\log p(-x)) &= \operatorname{trunc}_3\left(\sum_{\ell=1}^{3} \frac{(-1)^\ell}{\ell}(p(-x)-1)^\ell\right) \\ &= \sum_{i \in \{1,\dots,n\}} a_i x_i - \sum_{\substack{i,j \in \{1,\dots,n\} \\ i \leq j}} a_{ij} x_i x_j + \sum_{\substack{i,j,k \in \{1,\dots,n\} \\ i \leq j \leq k}} a_{ijk} x_i x_j x_k\end{aligned}$$

$$+ \frac{1}{2}\left(\sum_{i \in \{1,\dots,n\}} a_i x_i\right)^2 - \left(\sum_{i \in \{1,\dots,n\}} a_i x_i\right)\left(\sum_{\substack{i,j \in \{1,\dots,n\} \\ i \leq j}} a_{ij} x_i x_j\right) + \frac{1}{3}\left(\sum_{i \in \{1,\dots,n\}} a_i x_i\right)^3$$

It follows that

$$\begin{aligned} L_{p,d}(x_i) &= a_i, \\ \frac{1}{2} L_{p,d}(x_i^2) &= -a_{ii} + \frac{1}{2} a_i^2, \\ \frac{1}{3} L_{p,d}(x_i^3) &= a_{iii} - a_i a_{ii} + \frac{1}{3} a_i^3 \end{aligned}$$

for all $i \in \{1,\dots,n\}$,

$$\begin{aligned} L_{p,d}(x_i x_j) &= -a_{ij} + a_i a_j, \\ L_{p,d}(x_i^2 x_j) &= a_{iij} - a_i a_{ij} - a_j a_{ii} + a_i^2 a_j \end{aligned}$$

for all $i, j \in \{1,\dots,n\}$ with $i < j$, and

$$2L_{p,d}(x_i x_j x_k) = a_{ijk} - a_i a_{jk} - a_j a_{ik} - a_k a_{ij} + 2a_i a_j a_k$$

for all $i, j, k \in \{1,\dots,n\}$ with $i < j < k$.

Proposition 3.6. (a) If $p, q \in \mathbb{R}[[x]]$ satisfy $p(0) \neq 0 \neq q(0)$ and $d, e \in \mathbb{N}_0$, then

$$L_{pq,d+e} = L_{p,d} + L_{q,e}.$$

(b) If $p, q \in \mathbb{R}[x]$ satisfy $p(0) \neq 0 \neq q(0)$, then

$$L_{pq} = L_p + L_q.$$

Proof. Part (b) follows from (a) by observing that $\deg(pq) = \deg p + \deg q$ for all $p, q \in \mathbb{R}[x]$. To prove (a), we suppose WLOG $p(0) = 1 = q(0)$ and thus $(pq)(0) = 1$. By Definition 3.4, we then have to show that the following identity of formal power series holds:

$$-\log((pq)(-x)) = -\log(p(-x)) - \log(q(-x)).$$

This follows from Proposition 3.3(c). $\square$

Lemma 3.7. Let L be a linear form on the vector subspace V of $\mathbb{R}[x]$ generated by the monomials of degree k. Then

$$\sum_{\substack{\alpha\in\mathbb{N}_0^n\\|\alpha|=k}}\frac{1}{k}\binom{k}{\alpha}L((Ux)^\alpha)(Ux)^\alpha\in V$$

is the same polynomial for all orthogonal matrices $U\in\mathbb{R}^{n\times n}$.

Proof. Denote the i-th line of U by u_i (so that u_i is a row vector) and the j-th entry of u_i by u_{ij} for all $i,j\in\{1,\dots,n\}$. In the following, we often form the product of a row vector $u\in\mathbb{R}^n\subseteq\mathbb{R}[x]^n$ with the column vector $x\in\mathbb{R}[x]^n$ which is of course an element of $\mathbb{R}[x]$. Then

$$\begin{aligned}
\sum_{|\alpha|=k}\binom{k}{\alpha}L((Ux)^\alpha)(Ux)^\alpha &= \sum_{i_1,\dots,i_k=1}^{n}L(u_{i_1}x\cdots u_{i_k}x)u_{i_1}x\cdots u_{i_k}x\\
&= \sum_{i_1,\dots,i_k=1}^{n}\sum_{j_1,\dots,j_k=1}^{n}u_{i_1j_1}\cdots u_{i_kj_k}L(x_{j_1}\cdots x_{j_k})\sum_{\ell_1,\dots,\ell_k=1}^{n}u_{i_1\ell_1}\cdots u_{i_k\ell_k}x_{\ell_1}\cdots x_{\ell_k}\\
&= \sum_{\ell_1,\dots,\ell_k=1}^{n}\sum_{j_1,\dots,j_k=1}^{n}\left(\sum_{i_1,\dots,i_k=1}^{n}u_{i_1j_1}\cdots u_{i_kj_k}u_{i_1\ell_1}\cdots u_{i_k\ell_k}\right)L(x_{j_1}\cdots x_{j_k})x_{\ell_1}\cdots x_{\ell_k}\\
&= \sum_{\ell_1,\dots,\ell_k=1}^{n}\sum_{j_1,\dots,j_k=1}^{n}\underbrace{\left(\sum_{i_1=1}^{n}u_{i_1j_1}u_{i_1\ell_1}\right)}_{=\begin{cases}1 & \text{if } j_1=\ell_1\\ 0 & \text{otherwise}\end{cases}}\cdots\underbrace{\left(\sum_{i_k=1}^{n}u_{i_kj_k}u_{i_k\ell_k}\right)}_{=\begin{cases}1 & \text{if } j_k=\ell_k\\ 0 & \text{otherwise}\end{cases}}L(x_{j_1}\cdots x_{j_k})x_{\ell_1}\cdots x_{\ell_k}\\
&= \sum_{\ell_1,\dots,\ell_k=1}^{n}L(x_{\ell_1}\cdots x_{\ell_k})x_{\ell_1}\cdots x_{\ell_k}=\sum_{|\alpha|=k}\binom{k}{\alpha}L(x^\alpha)x^\alpha
\end{aligned}$$

for all orthogonal matrices $U\in\mathbb{R}^{n\times n}$. Multiplying with $\frac{1}{k}$ gives the result. □

Proposition 3.8. Let $U\in\mathbb{R}^{n\times n}$ be an orthogonal matrix.

(a) If $p\in\mathbb{R}[[x]]$ with $p(0)\neq0$ and $d\in\mathbb{N}_0$, then

$$L_{p(Ux),d}(q(Ux))=L_{p,d}(q).$$

(b) If $p\in\mathbb{R}[x]$ with $p(0)\neq0$, then

$$L_{p(Ux)}(q(Ux))=L_p(q).$$

Proof. Part (b) follows easily from (a) since $\deg(p(Ux))=\deg p$ for all $p\in\mathbb{R}[x]$. To prove (a), fix $p\in\mathbb{R}[[x]]$ with $p(0)\neq0$ and $d\in\mathbb{N}_0$. It suffices by linearity to prove $L_{p(Ux),d}((Ux)^\alpha)=L_{p,d}(x^\alpha)$ for all $\alpha\in\mathbb{N}_0^n$ with $\alpha\neq0$. WLOG $p(0)=1$. From Definition 3.2(b), one gets easily

$$\log(p(U(-x)))=(\log p)(-Ux).$$

This means by Definition 3.4 that

$$\sum_{\substack{\alpha\in\mathbb{N}_0^n\\ \alpha\neq0}}\frac{1}{|\alpha|}\binom{|\alpha|}{\alpha}L_{p(Ux),d}(x^\alpha)x^\alpha=\sum_{\substack{\alpha\in\mathbb{N}_0^n\\ \alpha\neq0}}\frac{1}{|\alpha|}\binom{|\alpha|}{\alpha}L_{p,d}(x^\alpha)(Ux)^\alpha.$$

We rewrite the left hand side by means of Lemma 3.7 to obtain

$$\sum_{\substack{\alpha\in\mathbb{N}_0^n\\ \alpha\neq 0}} \frac{1}{|\alpha|}\binom{|\alpha|}{\alpha} L_{p(Ux),d}((Ux)^\alpha)(Ux)^\alpha = \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ \alpha\neq 0}} \frac{1}{|\alpha|}\binom{|\alpha|}{\alpha} L_{p(x^\alpha),d}(Ux)^\alpha.$$

Substituting $U^T x$ for x, we finally get

$$\sum_{\substack{\alpha\in\mathbb{N}_0^n\\ \alpha\neq 0}} \frac{1}{|\alpha|}\binom{|\alpha|}{\alpha} L_{p(Ux),d}((Ux)^\alpha)x^\alpha = \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ \alpha\neq 0}} \frac{1}{|\alpha|}\binom{|\alpha|}{\alpha} L_{p,d}(x^\alpha)x^\alpha.$$

Comparing coefficients, we get the result. □

Definition 3.9. A polynomial is called *homogeneous* if all its monomials are of the same degree. If $p \in \mathbb{R}[x]$ is a polynomial of degree $d \in \mathbb{N}_0$, then the homogeneous polynomial

$$p^* := x_0^d\, p\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in \mathbb{R}[x_0, x]$$

is called its *homogenization* (with respect to x_0). In addition, we set

$$p^* := 0 \in \mathbb{R}[x_0, x].$$

The "shifted homogenization" of a polynomial used in Part (b) of the following lemma appears for real zero polynomials already in Brändén [?] and [NT].

Lemma 3.10. (a) For all $p \in \mathbb{R}[[x]]$ with $p(0) \neq 0$, the power series

$$q := (1+t)^d\, p\left(\frac{x_1}{1+t}, \dots, \frac{x_n}{1+t}\right) \in \mathbb{R}[[t, x]]$$

satisfies $q(0) \neq 0$ and

$$L_{q,d}(f) = L_{p,d}(f(1, x))$$

for all $d \in \mathbb{N}_0$ and $f \in \mathbb{R}[t, x]$.

(b) For all $p \in \mathbb{R}[x]$ with $p(0) \neq 0$, the polynomial

$$q := p^*(1+t, x_1, \dots, x_n) \in \mathbb{R}[t, x]$$

satisfies $q(0) \neq 0$ and

$$L_q(f) = L_p(f(1, x))$$

for all $f \in \mathbb{R}[t, x]$.

Proof. Part (b) follows from (a) by setting $d := \deg p$ and observing that $\deg p = \deg q$. To prove (a), we let $d \in \mathbb{N}_0$ and suppose WLOG $p(0) = 1$ so that $q(0) = 1$. By Definition 3.4, it remains to show the identity

$$-\log(q(-t, -x_1, \dots, -x_n)) = \sum_{\substack{k\in\mathbb{N}_0\\ \alpha\in\mathbb{N}_0^n\\ (k,\alpha)\neq 0}} \frac{1}{k+|\alpha|}\binom{k+|\alpha|}{k\ \alpha_1\ \dots\ \alpha_n} L_p(x^\alpha)t^k x^\alpha$$

of formal power series. By Proposition 3.3(c), it suffices to prove the identities

$$-\log((1-t)^d) = \sum_{k=1}^{\infty} \frac{1}{k} L_{p,d}(1)t^k$$

and

$$-\log\left(p\left(\frac{-x_1}{1-t},\dots,\frac{-x_n}{1-t}\right)\right)=\sum_{\substack{k\in\mathbb{N}_0\\ \alpha\in\mathbb{N}_0^n\\ \alpha\neq 0}}\frac{1}{k+|\alpha|}\binom{k+|\alpha|}{k\ \alpha_1\ \dots\ \alpha_n}L_{p,d}(x^\alpha)t^k x^\alpha$$

where one should note that the argument of the logarithm in the second identity is a power series since

$$\frac{1}{1-t}=\sum_{i=0}^{\infty}t^i.$$

Again by Proposition 3.3(c), we get $-\log((1-t)^d)=d\log(1-t)$. Together with $L_{p,d}(1)=d$ and Definition 3.2(b), this yields the first identity. To prove the second identity, we substitute

$$\frac{x_i}{1-t}=x_i\sum_{i=0}^{\infty}t^i$$

for x_i in the defining identity of $L_{p,d}$ from Definition 3.4 for each $i\in\{1,\dots,n\}$ to see that its left hand side equals

$$\sum_{\substack{\alpha\in\mathbb{N}_0^n\\ \alpha\neq 0}}\frac{1}{|\alpha|}\binom{|\alpha|}{\alpha}L_{p,d}(x^\alpha)\frac{x^\alpha}{(1-t)^{|\alpha|}}.$$

So it remains to show that for $\alpha\in\mathbb{N}_0^n$ with $\alpha\neq 0$, we have

$$\frac{\binom{|\alpha|}{\alpha}}{|\alpha|(1-t)^{|\alpha|}}=\sum_{k=0}^{\infty}\frac{1}{k+|\alpha|}\binom{k+|\alpha|}{k\ \alpha_1\ \dots\ \alpha_n}t^k.$$

Fix $\alpha\in\mathbb{N}_0^n$ with $\alpha\neq 0$. The multinomial coefficient on the right hand side divided by the one on the left hand side yields equals the binomial coefficient $\binom{k+|\alpha|}{k}$ which becomes $\binom{k+|\alpha|-1}{|\alpha|-1}$ when multiplied with $\frac{|\alpha|}{k+|\alpha|}$. So it remains to show that

$$\left(\sum_{i=0}^{\infty}t^i\right)^{|\alpha|}=\sum_{k=0}^{\infty}\binom{k+|\alpha|-1}{|\alpha|-1}t^k.$$

This is clear since the number of tuples of length $|\alpha|$ of nonnegative integers that sum up to k is the binomial coefficient on the right hand side. Indeed, choosing such a tuple amounts to partition $\{1,\dots,k\}$ into $|\alpha|$ discrete intervals. This in turn is equivalent to choosing $|\alpha|-1$ elements as separating landmarks in the set $\{1,\dots,k+|\alpha|-1\}$. □

Definition 3.11. Let $p\in\mathbb{R}[x]$ and $a\in\mathbb{R}^n$. Then we define the *a-transform* $p[a]$ of p by

$$p[a]:=p^*(1+a^Tx,x)\in\mathbb{R}[x].$$

In other words, $0[a]=0$ and if p is a polynomial of degree $d\in\mathbb{N}_0$, then

$$p[a]=(1+a^Tx)^d\,p\left(\frac{x_1}{1+a^Tx},\dots,\frac{x_n}{1+a^Tx}\right).$$

Remark 3.12. Let $p\in\mathbb{R}[x]$ and $a\in\mathbb{R}^n$.

(a) Of course, we have always $\deg(p[a]) \leq \deg p$. Unfortunately, the degree of $p[a]$ might sometimes be strictly smaller than the one of p, for example if $p = 1 + x_1 \in \mathbb{R}[x_1]$ and $a = -1 \in \mathbb{R} = \mathbb{R}^1$ where $p[a] = (1 - x_1) + x_1 = 1 \in \mathbb{R}[x_1]$. But the reader easily verifies that the degrees of p and $p[a]$ coincide if and only if the homogeneous polynomial $p^*(a^T x, x)$ is not the zero polynomial.

(b) It is an easy exercise to show that

$$p[a][b] = p[a + b]$$

in the case where $\deg p = \deg(p[a])$.

Proposition 3.13. Suppose $p \in \mathbb{R}[x]$ is a real zero polynomial and $a \in \mathbb{R}^n$. Then $p[a]$ is again a real zero polynomial.

Proof. Suppose $b \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ such that $p[a](\lambda b) = 0$. We have to show $\lambda \in \mathbb{R}$. If $1 + \lambda a^T b = 0$, then $a^T b \neq 0$ and thus $\lambda = -\frac{1}{a^T b} \in \mathbb{R}$. Suppose therefore $1 + \lambda a^T b \neq 0$. Then $p\left(\frac{\lambda}{1+\lambda a^T b} b\right) = 0$ and thus $c := \frac{\lambda}{1+\lambda a^T b} \in \mathbb{R}$ since p is a real zero polynomial. If $c = 0$, then $\lambda = 0 \in \mathbb{R}$ and we are done. Suppose therefore $c \neq 0$. Then $\lambda = c + \lambda c a^T b$ and hence $\lambda(1 - c a^T b) = c \neq 0$ which again implies $\lambda \in \mathbb{R}$. $\square$

Proposition 3.14. Let $a \in \mathbb{R}^n$, $p \in \mathbb{R}[x]$ with $p(0) \neq 0$ and $d \in \mathbb{N}_0$. Then

$$L_{p[a],d}(f(x)) = L_{p,d}(f(x + a))$$

for all $f \in \mathbb{R}[x]$.

Proof. WLOG $p(0) = 1$ and thus $p[a](0) = 1$. For the duration of this proof, we denote by $\preceq$ the the partial order on $\mathbb{N}_0^n$ which stand for the componentwise natural order, i.e.,

$$\alpha \preceq \beta :\iff \forall i \in \{1, \ldots, n\} : \alpha_i \leq \beta_i$$

for $\alpha, \beta \in \mathbb{N}_0^n$. From Lemma 3.10, we know that

$$-\log(p^*(1 - t, -x_1, \ldots, -x_n)) = \sum_{\substack{k \in \mathbb{N}_0 \\ \alpha \in \mathbb{N}_0^n \\ (k,\alpha) \neq 0}} \frac{1}{k + |\alpha|} \binom{k + |\alpha|}{k \ \alpha_1 \ \ldots \ \alpha_n} L_{p,d}(x^\alpha) t^k x^\alpha.$$

Substituting $a^T x$ for t in this identity, yields

$$\begin{aligned}
-\log(p[a](-x)) &= \sum_{\substack{k\in\mathbb{N}_0\\ \alpha\in\mathbb{N}_0^n\\ (k,\alpha)\neq 0}} \frac{1}{k+|\alpha|}\binom{k+|\alpha|}{k\ \alpha_1\ \ldots\ \alpha_n} L_{p,d}(x^\alpha)(a^Tx)^k x^\alpha \\
&= \sum_{\substack{k\in\mathbb{N}_0\\ \alpha\in\mathbb{N}_0^n\\ (k,\alpha)\neq 0}} \frac{1}{k+|\alpha|}\binom{k+|\alpha|}{k\ \alpha_1\ \ldots\ \alpha_n} L_{p,d}(x^\alpha)\left(\sum_{\substack{\beta\in\mathbb{N}_0^n\\ |\beta|=k}}\binom{k}{\beta}a^\beta x^\beta\right) x^\alpha \\
&= \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ (\alpha,\beta)\neq 0}} \frac{1}{|\alpha|+|\beta|}\binom{|\alpha|+|\beta|}{|\beta|\ \alpha_1\ \ldots\ \alpha_n} L_{p,d}(x^\alpha)\binom{|\beta|}{\beta}a^\beta x^{\alpha+\beta} \\
&= \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ (\alpha,\beta)\neq 0}} \frac{1}{|\alpha|+|\beta|}\binom{|\alpha|+|\beta|}{\alpha\quad\beta} L_{p,d}(x^\alpha)a^\beta x^{\alpha+\beta} \\
&= \sum_{\substack{\gamma\in\mathbb{N}_0^n\\ \gamma\neq 0}}\sum_{\substack{\alpha\in\mathbb{N}_0^n\\ \alpha\preceq\gamma}} \frac{1}{|\gamma|}\binom{|\gamma|}{\alpha\quad\gamma-\alpha} L_{p,d}(x^\alpha)a^{\gamma-\alpha}x^\gamma \\
&= \sum_{\substack{\gamma\in\mathbb{N}_0^n\\ \gamma\neq 0}} \frac{1}{|\gamma|}\binom{|\gamma|}{\gamma}\left(\sum_{\substack{\alpha\in\mathbb{N}_0^n\\ \alpha\preceq\gamma}}\binom{\gamma_1}{\alpha_1}\cdots\binom{\gamma_n}{\alpha_n}L_{p,d}(x^\alpha)a^{\gamma-\alpha}\right)x^\gamma \\
&= \sum_{\substack{\gamma\in\mathbb{N}_0^n\\ \gamma\neq 0}} \frac{1}{|\gamma|}\binom{|\gamma|}{\gamma}L_{p,d}\left(\prod_{i=1}^n\sum_{\alpha_i=1}^{\gamma_i}\binom{\gamma_i}{\alpha_i}{x_i}^{\alpha_i}{a_i}^{\gamma_i-\alpha_i}\right)x^\gamma \\
&= \sum_{\substack{\gamma\in\mathbb{N}_0^n\\ \gamma\neq 0}} \frac{1}{|\gamma|}\binom{|\gamma|}{\gamma}L_{p,d}((x+a)^\gamma)x^\gamma.
\end{aligned}$$

This implies $L_{p[a],d}(x^\alpha) = L_{p,d}((x+a)^\alpha)$ for all $\alpha\in\mathbb{N}_0^n$ with $\alpha\neq 0$. By linearity, this shows the claim. □

3.2. **Linear forms and traces.**

Lemma 3.15. Let

$$\sum_{k=1}^\infty a_k t^k \in \mathbb{C}[[t]] \qquad (a_1,a_2,\ldots\in\mathbb{C})$$

be a univariate power series with positive radius of convergence. Let $p\in\mathbb{C}[x]$ be a polynomial with $p(0)=0$. Then there is some $\varepsilon>0$ such that for each $z\in\mathbb{C}$ with $|z|<\varepsilon$, the series

$$\sum_{k=1}^\infty a_k p(z)^k$$

is absolutely convergent even after fully expanding $a_k p(z)^k$ into the obvious sum of m^k many terms where m is the number of monomials in p.

Proof. We have to bound the finite partial sums of the absolute values of the individual terms from above (see for example [T1, Definition 8.2.4] or [Rui, §I.1]). Write $p = t_1 + \ldots + t_m$ where each t_i involves only one monomial of p. We have to find $\varepsilon > 0$ and $C \in \mathbb{R}$ such that for each $z \in \mathbb{C}$ with $|z| < \varepsilon$ and for each $\ell \in \mathbb{N}$, we have

$$\sum_{k=1}^{\ell} \sum_{i_1=1}^{m} \ldots \sum_{i_k=1}^{m} |a_k t_{i_1}(z) \cdots t_{i_k}(z)| \leq C.$$

WLOG $m > 0$. Let $r > 0$ denote the radius of convergence of the univariate power series. Choose $\varrho \in \mathbb{R}$ with $0 < \varrho < r$. As $p(0) = 0$, each t_i vanishes at the origin. By continuity, we can choose $\varepsilon > 0$ such that

$$|t_i(z)| \leq \frac{\varrho}{m}$$

for all $i \in \{1, \ldots, m\}$ and $z \in \mathbb{C}$ with $|z| < \varepsilon$. Since a univariate power series converges absolutely inside the radius of convergence [T2, Theorem 4.1.6(b)], we can set $C := \sum_{k=1}^{\infty} |a_k| \varrho^k < \infty$. For each $\ell \in \mathbb{N}$ and $z \in \mathbb{C}$ with $|z| < \varepsilon$, we have

$$\sum_{k=1}^{\ell} \sum_{i_1=1}^{m} \ldots \sum_{i_k=1}^{m} |a_k t_{i_1} \cdots t_{i_k}| \leq \sum_{k=1}^{\ell} m^k |a_k| \left(\frac{\varrho}{m}\right)^k = \sum_{k=1}^{\ell} |a_k| \varrho^k \leq C.$$

□

Definition 3.16. Let $A_1, \ldots, A_n \in \mathbb{C}^{d \times d}$ and $\alpha \in \mathbb{N}_0^n$. The *$\alpha$-Hurwitz product* of $A_1, \ldots, A_n$ is the matrix that arises as follows: First, form all words in n letters where the i-th letter appears exactly α_i times. Then turn each word into a product of matrices by substituting A_i for the i-th letter. Finally, sum up all matrices that arise in this way. Formally, we can define it as

$$\operatorname{hur}_\alpha(A_1, \ldots, A_n) := \sum_{\substack{f \colon \{1, \ldots, |\alpha|\} \to \{1, \ldots, n\} \\ \forall i \in \{1, \ldots, n\} : \# f^{-1}(i) = \alpha_i}} A_{f(1)} \cdots A_{f(|\alpha|)} \in \mathbb{C}^{d \times d}.$$

In particular, $\operatorname{hur}_0(A_1, \ldots, A_n) = I_d$.

Proposition 3.17. Suppose $d \in \mathbb{N}_0$ and $A_1, \ldots, A_n \in \mathbb{C}^{d \times d}$ are hermitian. Then

$$p := \det(I_d + x_1 A_1 + \ldots + x_n A_n) \in \mathbb{R}[x]$$

and

$$L_{p,d}(x^\alpha) = \frac{1}{\binom{|\alpha|}{\alpha}} \operatorname{tr}(\operatorname{hur}_\alpha(A_1, \ldots, A_n))$$

for all $\alpha \in \mathbb{N}_0^n$.

Proof. For $q \in \mathbb{C}[x]$, we denote by $q^* \in \mathbb{C}[x]$ the polynomial which arises from q by applying the complex conjugation to the coefficients. We have

$$\begin{aligned} p^* &= (\det(I_d + x_1 A_1 + \ldots + x_n A_n))^* = (\det((I_d + x_1 A_1 + \ldots + x_n A_n)^T))^* \\ &= (\det(I_d + x_1 A_1^T + \ldots + x_n A_n^T))^* = \det(I_d + x_1 A_1^* + \ldots + x_n A_n^*) \\ &= \det(I_d + x_1 A_1 + \ldots + x_n A_n) = p \end{aligned}$$

and therefore $p \in \mathbb{R}[x]$. It is easy to see that Hurwitz products of hermitian matrices are again hermitian and therefore have real diagonal entries and henceforth real trace.

It is clear that $L_{p,d}(1) = d = \operatorname{tr}(I_d) = \operatorname{tr}(\operatorname{hur}_0(A_1,\dots,A_n))$. By Definition 3.4, it remains to show that

$$(*) \qquad -\log(p(-x)) = \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ \alpha\neq 0}} \frac{1}{|\alpha|}\operatorname{tr}(\operatorname{hur}_\alpha(A_1,\dots,A_n))x^\alpha.$$

The real multivariate power series on both sides converge absolutely in a neighborhood of the origin in $\mathbb{R}^n$. For the left hand side this follows from Lemma 3.15 by recollecting terms belonging to the same monomial. For the right hand side, we argue as follows: The number of words of length k in n letters is n^k. If the entries of each A_i are bounded in absolute value by $c > 0$, then the entries of a product of the A_i with k many factors are bounded in absolute value by $d^{k-1}c^k$. Then the trace of such a product is bounded by $(dc)^k$. Hence we get

$$\sum_{\substack{\alpha\in\mathbb{N}_0^n\\ |\alpha|=k}} \left|\frac{1}{|\alpha|}\operatorname{hur}_\alpha(A_1,\dots,A_n)a^\alpha\right| \leq (cdn)^k\|a\|_\infty^k \leq \left(\frac{1}{2}\right)^k$$

for all $a\in\mathbb{R}^n$ with $\|a\|_\infty \leq \frac{1}{2cdn}$.

By the identity theorem for multivariate real power series [Rui, Proposition 2.9], it suffices to show that both series in $(*)$ converge absolutely to the same value in a neighborhood of the origin in $\mathbb{C}^n$. It is a subtle issue that uses Lemma 3.15 and rearrangement of absolutely convergent series (cf. Proposition [Rui, Proposition 1.6]) to show that for all a in a neighborhood of the origin in $\mathbb{R}^n$, the left hand side of $(*)$ evaluated at a (i.e., $(-\log(p(-x)))(a)$ equals $-\log(p(-a))$ where the first log stands for the operation on power series defined in Definition 3.2 and the second one for the usual real logarithm. On the other hand, the right hand side of $(*)$ evaluates at a from a small neighborhood of the origin to

$$\sum_{\substack{\alpha\in\mathbb{N}_0^n\\ \alpha\neq 0}} \frac{1}{|\alpha|}\operatorname{tr}(\operatorname{hur}_\alpha(a_1A_1,\dots,a_nA_n)) = \sum_{k=1}^\infty \frac{1}{k}\operatorname{tr}((a_1A_1+\ldots+a_nA_n)^k).$$

It now suffices to fix $a\in\mathbb{R}^n$ such that the hermitian matrix

$$B := a_1A_1+\ldots+a_nA_n \in \mathbb{C}^{d\times d}$$

is of operator norm strictly less than 1 (or equivalently has all eigenvalues in the open real interval $(-1,1)$) and to show that

$$-\log(\det(I_n - B)) = \sum_{k=1}^\infty \frac{1}{k}\operatorname{tr}(B^k).$$

Since the operator norm is sub-multiplicative, the matrix $C := \sum_{k=1}^\infty \frac{1}{k}B^k \in \mathbb{C}^{d\times d}$ exists. Obviously, C is hermitian and its eigenvalues, listed according to their algebraic multiplicity, arise from the eigenvalues of $I_n - B$ by taking minus the real logarithm. Since determinant and trace are the product and sum, respectively, of the eigenvalues counted with algebraic multiplicity, we thus get the result. □

The traces of Hurwitz products appearing in Proposition 3.17 are in general hard to deal with. It is an easy but absolutely crucial observation that this is different for Hurwitz products with up to three factors.

Corollary 3.18. Suppose $d \in \mathbb{N}_0$ and $A_1, \dots, A_n \in \mathbb{C}^{d\times d}$ are hermitian. Set

$$p := \det(I_d + x_1A_1 + \ldots + x_nA_n) \in \mathbb{R}[x].$$

Then

$$\begin{aligned}
L_{p,d}(1) &= \operatorname{tr}(I_d),\\
L_{p,d}(x_i) &= \operatorname{tr}(A_i),\\
L_{p,d}(x_ix_j) &= \operatorname{tr}(A_iA_j) = \operatorname{tr}(A_jA_i) \qquad \text{and}\\
L_{p,d}(x_ix_jx_k) &= \operatorname{Re}\operatorname{tr}(A_iA_jA_k) = \operatorname{Re}\operatorname{tr}(A_iA_kA_j) = \operatorname{Re}\operatorname{tr}(A_jA_iA_k)\\
&= \operatorname{Re}\operatorname{tr}(A_jA_kA_i) = \operatorname{Re}\operatorname{tr}(A_kA_iA_j) = \operatorname{Re}\operatorname{tr}(A_kA_jA_i)
\end{aligned}$$

for all $i, j, k \in \{1, \dots, n\}$.

Proof. The first three statements are trivial. For the last statement, note that

$$(\operatorname{tr}(ABC))^* = (\operatorname{tr}((ABC)^T))^* = (\operatorname{tr}(C^TB^TA^T))^* = \operatorname{tr}(C^*B^*A^*) = \operatorname{tr}(CBA)$$

for all hermitian $A, B, C \in \mathbb{C}^{d\times d}$. □

3.3. **Relaxing hyperbolic programs.**

Definition 3.19. Let $p \in \mathbb{R}[[x]]$ be a power series with $p(0) \neq 0$ and $d \in \mathbb{N}_0$. Consider the symmetric matrices

$$A_0 := \begin{pmatrix} L_{p,d}(1) & L_{p,d}(x_1) & \ldots & L_{p,d}(x_n)\\ L_{p,d}(x_1) & L_{p,d}(x_1^2) & \ldots & L_{p,d}(x_1x_n)\\ \vdots & \vdots & & \vdots\\ L_{p,d}(x_n) & L_{p,d}(x_1x_n) & \ldots & L_{p,d}(x_n^2)\end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

and

$$A_i := \begin{pmatrix} L_{p,d}(x_i) & L_{p,d}(x_ix_1) & \ldots & L_{p,d}(x_ix_n)\\ L_{p,d}(x_ix_1) & L_{p,d}(x_ix_1^2) & \ldots & L_{p,d}(x_ix_1x_n)\\ \vdots & \vdots & & \vdots\\ L_{p,d}(x_ix_n) & L_{p,d}(x_ix_1x_n) & \ldots & L_{p,d}(x_ix_n^2)\end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

for $i \in \{1, \dots, n\}$.

(a) We call the linear matrix polynomial

$$M_{p,d} := A_0 + x_1A_1 + \ldots + x_nA_n \in \mathbb{R}[x]^{(n+1)\times(n+1)}$$

the *pencil associated to p with respect to the virtual degree d* and

$$S_d(p) := \{a \in \mathbb{R}^n \mid M_{p,d}(a) \succeq 0\}$$

the *spectrahedron associated to p with respect to the virtual degree d*.

(b) In the case where p is a polynomial, we call

$$M_p := M_{p,\deg f}$$

the *pencil associated to p* and

$$S(p) := S_{\deg f}(p)$$

the *spectrahedron associated to p*.

(c) We call the linear matrix polynomial

$$M_{p,\infty} \in \mathbb{R}[x]^{n\times n}$$

that arises from $M_{p,d}$ (for no matter what $d \in \mathbb{N}_0$) by deleting the first row and column the *pencil associated to p with respect to infinite virtual degree* and

$$S_\infty(p) := \{a \in \mathbb{R}^n \mid M_{p,\infty}(a) \succeq 0\}$$

the *spectrahedron associated to p with respect to infinite virtual degree*.

Remark 3.20. Let $p \in \mathbb{R}[[x]]$ be a power series with $p(0) \neq 0$. Then

$$S_0(p) \subseteq S_1(p) \subseteq S_2(p) \subseteq S_3(p) \subseteq S_4(p) \subseteq \ldots \subseteq S_\infty(p).$$

Remark 3.21. Let $p \in \mathbb{R}[x]$ be a polynomial with $p(0) \neq 0$. Note that M_p and therefore $S(p)$ depend only on the cubic part $\operatorname{trunc}_3 p$ of p. Indeed, if one assumes moreover that $p(0) = 1$ then this is a polynomial dependance on the corresponding coefficients of p, more exactly a cubic one which could be written down explicitly by the expressions of Example 3.5 for the values of L_p on the monomials of degree at most 3.

Lemma 3.22. Let $p \in \mathbb{R}[x]$ be a power series with $p(0) \neq 0$, $d \in \mathbb{N}_0$, $a \in \mathbb{R}^n$ and

$$v = \begin{pmatrix} v_0 & v_1 & \ldots & v_n \end{pmatrix}^T \in \mathbb{R}^{n+1}.$$

Then $v^T M_{p,d}(a) v = L_{p,d}((v_0 + v_1x_1 + \ldots + v_nx_n)^2(1 + a_1x_1 + \ldots + a_nx_n))$.

Proof. For the moment denote $x_0 := 1$ an $a_0 := 1$. Then

$$\begin{aligned} v^T M_{p,d}(a) v &= \sum_{i=0}^n \sum_{j=0}^n v_i v_j \sum_{k=0}^n a_k L_{p,d}(x_i x_j x_k) \\ &= L_{p,d}\left(\left(\sum_{i=0}^n v_i x_i\right)\left(\sum_{j=0}^n v_j x_j\right)\left(\sum_{k=0}^n a_k x_k\right)\right). \end{aligned}$$

□

Lemma 3.23. Suppose $U \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and consider the orthogonal matrix

$$\widetilde{U} := \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

(a) If $p \in \mathbb{R}[[x]]$ is a power series with $p(0) \neq 0$ and $d \in \mathbb{N}_0$, then

$$M_{p(Ux),d} = \widetilde{U}^T M_{p,d}(Ux)\widetilde{U}.$$

(b) If $p \in \mathbb{R}[x]$ is a polynomial with $p(0) \neq 0$, then

$$M_{p(Ux)} = \widetilde{U}^T M_p(Ux)\widetilde{U}.$$

(c) If $p \in \mathbb{R}[[x]]$ is a power series with $p(0) \neq 0$, then

$$M_{p(Ux),\infty} = U^T M_{p,\infty}(Ux) U.$$

Proof. Part (c) is immediate from (a). Part (b) follows from (a) by observing that $\deg(p(Ux)) = \deg p$ for all polynomials $p \in \mathbb{R}[x]$. To prove (a), we let $p \in \mathbb{R}[[x]]$ be a power series with $p(0) \neq 0$ and $d \in \mathbb{N}_0$. We can rewrite the claim as

$$\widetilde{U} M_{p(Ux),d} \widetilde{U}^T = M_{p,d}(Ux)$$

which in turn is equivalent to

$$\widetilde{U}M_{p(Ux),d}(U^Tx)\widetilde{U}^T = M_{p,d}$$

by the automorphisms of the power series ring $\mathbb{R}[x]$ given by $x \mapsto Ux$ and $x \mapsto U^Tx$. For each $v \in \mathbb{R}^n$, we denote by $\widetilde{v} \in \mathbb{R}^{n+1}$ the vector that arises from v by prepending 1. By continuity, homogeneity and the identity theorem for multivariate polynomials, it suffices to show that

$$\widetilde{v}^T\widetilde{U}M_{p(Ux),d}(U^Ta)\widetilde{U}^T\widetilde{v} = \widetilde{v}^TM_{p,d}(a)\widetilde{v}$$

for all $a, v \in \mathbb{R}^n$. By Lemma 3.22, this is equivalent to

$$L_{p(Ux),d}((1+(U^Tv)^Tx)^2(1+(U^Ta)^Tx)) = L_{p,d}((1+v^Tx)^2(1+a^Tx))$$

for all $a, v \in \mathbb{R}^n$ which follows easily from Proposition 3.8(a) after rewriting the left hand side as $L_{p(Ux),d}((1+v^TUx)^2(1+a^TUx))$. □

Proposition 3.24. Let $p \in \mathbb{R}[x]$ with $p(0) \neq 0$ and $U \in \mathbb{R}^{n\times n}$ an orthogonal matrix. Then

$$\begin{aligned} C(p(Ux)) &= \{U^Ta \mid a \in C(p)\}, \\ S(p(Ux)) &= \{U^Ta \mid a \in S(p)\} \qquad \text{and} \\ S_d(p(Ux)) &= \{U^Ta \mid a \in S_d(p)\} \end{aligned}$$

for all $d \in \mathbb{N}_0 \cup \{\infty\}$.

Proof. We have

$$\begin{aligned} C(p(Ux)) &= \{a \in \mathbb{R}^n \mid \forall \lambda \in [0,1) : p(U(\lambda a)) \neq 0\} \\ &= \{a \in \mathbb{R}^n \mid \forall \lambda \in [0,1) : p(\lambda Ua) \neq 0\} \\ &= \{U^Ta \in \mathbb{R}^n \mid \forall \lambda \in [0,1) : p(\lambda a) \neq 0\} = \{U^Tx \mid x \in C(p)\} \end{aligned}$$

and using Lemma 3.23(b),

$$\begin{aligned} S(p(Ux)) &= \{a \in \mathbb{R}^n \mid M_{p(Ux)}(a) \succeq 0\} \\ &= \{a \in \mathbb{R}^n \mid M_p(Ua) \succeq 0\} \\ &= \{U^Ta \in \mathbb{R}^n \mid M_p(a) \succeq 0\} \\ &= \{U^Ta \in \mathbb{R}^n \mid a \in S(p)\}. \end{aligned}$$

The last statement follows in a similar way from Lemma 3.23(a). □

Lemma 3.25. Let $p \in \mathbb{R}[x]$ be a polynomial with $p(0) \neq 0$ and set $d := \deg p$. Then

$$P := \begin{pmatrix} 1 & a^T \\ 0 & I_n \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

is invertible and

$$M_{p[a],d} = P^T(M_p + a^TxM_p(0))P.$$

Proof. For each $v \in \mathbb{R}^n$, we denote by $\widetilde{v} \in \mathbb{R}^{n+1}$ the vector that arises from v by prepending 1. By continuity, homogeneity and the identity theorem for multivariate polynomials, it suffices to show that

$$\widetilde{v}^TM_{p[a],d}(b)\widetilde{v} = \widetilde{v}^TP^T(M_p(b) + a^TbM_p(0))P\widetilde{v}$$

for all $b, v \in \mathbb{R}^n$. Fixing $b, v \in \mathbb{R}^n$ and setting $w := P\widetilde{v} = \begin{pmatrix} 1 + a^T v \\ v \end{pmatrix} \in \mathbb{R}^{n+1}$, this amounts to show

$$\widetilde{v}^T M_{p[a],d}(b)\widetilde{v} = w^T M_p(b)w + (a^T b)w^T M_p(0)w.$$

Applying Lemma 3.22, this is equivalent to

$$L_{p[a],d}((1+v^T x)^2(1+b^T x)) = \\ L_p(((1+a^T v) + v^T x)^2(1+b^T x)) + a^T b L_p(((1+a^T v) + v^T x)^2)$$

for all $b, v \in \mathbb{R}^n$ which follows easily from Proposition 3.14 after rewriting the left hand side as $L_p((1+v^T(x+a))^2(1+b^T(x+a)))$. □

Lemma 3.26. Suppose $m, n \in \mathbb{N}_0$ with $m \leq n$ and $q \in \mathbb{R}[x]$ with $q(0) \neq 0$. Set

$$r := q(x_1, \ldots, x_m, 0, \ldots, 0) \in \mathbb{R}[x_1, \ldots, x_m].$$

Then

(a) $L_{q,d}(p) = L_{r,d}(p)$ for all $p \in \mathbb{R}[x_1, \ldots, x_m]$ and $d \in \mathbb{N}_0$
(b) $\{a \in \mathbb{R}^m \mid (a, 0, \ldots, 0) \in C(q)\} = C(r)$
(c) $\{a \in \mathbb{R}^m \mid (a, 0, \ldots, 0) \in S_d(q)\} \subseteq S_d(r)$ for all $d \in \mathbb{N}_0 \cup \{\infty\}$

Proof. (a) By linearity, it suffices to consider the case where p is a monomial. If $p = 1$, then $L_{q,d}(p) = d = L_{r,d}(p)$. It remains to show that $L_{q,d}(x^\alpha) = L_{r,d}(x^\alpha)$ for all $a \in \mathbb{N}_0^m$ with $\alpha \neq 0$. But this follows from Definitions 3.4 and 3.2(b) since the power series $\log r$ arises from the power series $\log q$ by substituting the variables $x_{m+1}, \ldots, x_n$ with 0.

(b) is clear.

(c) follows from (a) together with Lemma 3.22. □

Proposition 3.27. Fix $d \in \mathbb{N}_0$. Then

$$(A, B) \mapsto \operatorname{tr}(AB)$$

is a scalar product on the real vector space of hermitian matrices in $\mathbb{C}^{d\times d}$. In particular, $\operatorname{tr}(AB) \in \mathbb{R}$ for all hermitian $A, B \in \mathbb{C}^{d\times d}$.

Proof. Identifying each matrix of size d with a "long" vector of size d^2 by reading its entries in the usual way, the scalar product is induced by the usual complex scalar product on $\mathbb{C}^d$. Since all diagonal entries of a hermitian matrix are real and all other entries have the opposite imaginary part of its mirror entry, the claim easily follows. □

Lemma 3.28. Suppose $d \in \mathbb{N}_0$ and $A_1, \ldots, A_n \in \mathbb{C}^{d\times d}$ are hermitian. Set

$$p := \det(I_d + x_1 A_1 + \ldots + x_n A_n) \in \mathbb{R}[x].$$

For all $a \in \mathbb{R}^n$ and

$$v = \begin{pmatrix} v_0 & v_1 & \ldots & v_n \end{pmatrix}^T \in \mathbb{R}^{n+1},$$

we then have

$$v^T M_{p,d}(a)v = \operatorname{tr}((v_0 I_d + v_1 A_1 + \ldots + v_n A_n)^2(I_d + a_1 A_1 + \ldots + a_n A_n)).$$

Proof. Corollary 3.18, Lemma 3.22 and Proposition 3.27. □

Since M^2 is hermitian for each hermitian $M \in \mathbb{C}^{d\times d}$, Proposition 3.27 shows that the traces occurring in the next definition are real. Moreover, since M^2 is even psd for each hermitian $M \in \mathbb{C}^{d\times d}$ and the trace of a product of two psd matrices nonnegative, we see that the two occurrences of " $\implies$ " could equivalently be replaced by " $\iff$ " in the next definition.

Definition 3.29. We call U *perfect* if it is a subset of $\{A \in \mathbb{C}^{d\times d} \mid A \text{ hermitian}\}$ that satisfies

$$\forall A \in U : ((\forall M \in U : \operatorname{tr}(M^2A) \geq 0) \implies A \succeq 0).$$

We call (U, V) an *admissible couple* if $U \subseteq V \subseteq \{A \in \mathbb{C}^{d\times d} \mid A \text{ hermitian}\}$ and

$$\forall A \in U : ((\forall M \in V : \operatorname{tr}(M^2A) \geq 0) \implies A \succeq 0).$$

Remark 3.30. (a) Let $U \subseteq \mathbb{C}^{d\times d}$ be perfect and $k \in \mathbb{N}_0$. Then

$$\left\{ \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A \end{pmatrix} \in \mathbb{C}^{(kd)\times(kd)} \;\middle|\; A \in U \right\}$$

is again perfect.

(b) Let $U \subseteq \mathbb{C}^{d\times d}$ and $V \subseteq \mathbb{C}^{e\times e}$ be perfect and suppose $0 \in U$ and $0 \in V$. Then

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathbb{C}^{(d+e)\times(d+e)} \;\middle|\; A \in U, B \in V \right\}$$

is again perfect.

(c) Let $U \subseteq \mathbb{C}^{d\times d}$ be perfect and $Q \in \mathbb{C}^{d\times d}$ be a unitary matrix (e.g., a permutation matrix). Then

$$\{Q^*AQ \mid A \in U\}$$

is again perfect.

Remark 3.31. The following is an easy exercise that we leave to the reader:

(a) The following sets are perfect:
- $\{\lambda I_d \mid \lambda \in \mathbb{R}\}$
- $\{A \in \mathbb{R}^{d\times d} \mid A \text{ diagonal}\}$
- $\{A \in \mathbb{R}^{d\times d} \mid A \text{ symmetric}\}$
- $\{A \in \mathbb{C}^{d\times d} \mid A \text{ hermitian}\}$

(b) If V is a perfect set and U is contained in it, then (U, V) is an admissible couple.

Example 3.32. Let $A \in \mathbb{C}^{d\times d}$ be hermitian. We claim that the real span of

$$I_d, A, A^2, \dots, A^{d-1}$$

is perfect. Indeed, by conjugating A with a suitable unitary matrix, one easily reduces to the case where A is a diagonal matrix with diagonal entries $a = (a_1, \dots, a_d) \in \mathbb{R}^n$. By conjugating it once more with a permutation matrix, we can moreover suppose that the first n entries $a_1, \dots, a_n$ of a are pairwise distinct and all other entries $a_{n+1}, \dots, a_d$ are repetitions of entries of $a_1, \dots, a_n$. Consider now the Vandermonde matrix

$$H := \begin{pmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{d-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{d-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_d & a_d^2 & \dots & a_d^{d-1} \end{pmatrix} \in \mathbb{R}^{d \times d}$$

whose columns are the diagonals of the diagonal matrices $I_d, A, \dots, A^{d-1}$. The top left $d \times d$ submatrix is invertible since it is again a Vandermonde matrix with pairwise different rows. Hence the projection of the column space of H on the first d components is $\mathbb{R}^d$. The rows $n+1$ to d of H are repetitions of the first n rows of H. The entries of each element in the column space follow the same pattern. Hence it suffices to apply Remark 3.30(a) n-times to the perfect set $\mathbb{R} = \mathbb{R}^{1 \times 1}$ (each time with a possibly different appropriate number of repetitions k), then use several times Remark 3.30(b) and finally apply Remark 3.30(c) with a suitable permutation matrix.

Proposition 3.33. Suppose $d \in \mathbb{N}_0$, $A_1, \dots, A_n \in \mathbb{C}^{d \times d}$,

$$U := \{v_0 I_d + v_1 A_1 + \dots + v_n A_n \mid v_0, v_1, \dots, v_n \in \mathbb{R}\},$$
$$U_\infty := \{v_1 A_1 + \dots + v_n A_n \mid v_1, \dots, v_n \in \mathbb{R}\}$$

and (U, V) is an admissible couple (in particular, each A_i is hermitian). Set

$$p := \det(I_d + x_1 A_1 + \dots + x_n A_n) \in \mathbb{R}[x].$$

(a) We have

$$C(p) = \{a \in \mathbb{R}^n \mid \forall M \in V : \operatorname{tr}(M^2(I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0\},$$
$$S_d(p) = \{a \in \mathbb{R}^n \mid \forall M \in U : \operatorname{tr}(M^2(I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0\} \quad \text{and}$$
$$S_\infty(p) = \{a \in \mathbb{R}^n \mid \forall M \in U_\infty : \operatorname{tr}(M^2(I_d + a_1 A_1 + \dots + a_n A_n)) \geq 0\}.$$

(b) $C(p) \subseteq S_d(p) \subseteq S_\infty(p)$
(c) If U is perfect, then $C(p) = S_d(p)$.
(d) If U_∞ is perfect, then $C(p) = S_d(p) = S_\infty(p)$.

Proof. The first statement in (a) follows directly from Proposition 2.12 together with Definition 3.29. The remaining statements of (a) follow easily from Lemma 3.28 and Definition 3.19. Statement (b) is a direct consequence of (a) since $U_\infty \subseteq U \subseteq V$. Part (c) and (d) now follow directly from Definition 3.29. □

Example 3.34. Let $A_1, \dots, A_n \in \mathbb{R}^{d \times d}$ by symmetric and consider

$$p := \det(I_d + x_1 A_1 + \dots + x_n A_n) \in \mathbb{R}[x]$$

which has degree at most d.

(a) Suppose the A_i together with I_d generate the vector space of all real symmetric matrices of size d, then $C(p) = S_d(p)$.
(b) Suppose the A_i themselves generate the vector space of all real symmetric matrices of size d, then even $C(p) = S_\infty(p)$.

For real zero polynomials having a determinantal representation as in Proposition 3.33 whose size equals their degree, the following is an immediate consequence of Proposition 3.33. However, in the general case we have to argue in a much more subtle way. Actually, this is the first place in this article where we wouldn't know how to avoid the Helton-Vinnikov theorem (in form of Corollary 2.8).

Theorem 3.35. Let p be a real zero polynomial. Then $C(p) \subseteq S(p)$.

Proof. WLOG $p(0) = 1$. For $n \leq 2$, the claim follows from Proposition 3.33(b) where we use the Helton-Vinnikov Corollary 2.8 for $n = 2$. We now suppose $n > 2$ and reduce it to the already proven case $n = 2$. Let $a \in C(p)$. We have to show $M_p(a) \succeq 0$. By continuity and homogeneity, it suffices to show

$$\begin{pmatrix} 1 & v^T \end{pmatrix} M_p(a) \begin{pmatrix} 1 \\ v \end{pmatrix} \geq 0$$

for all $v \in \mathbb{R}^n$. By Lemma 3.22, this is equivalent to

$$L_p((1 + v^T x)^2 (1 + a^T x)) \geq 0.$$

To prove this, choose an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ such that $w := U^T v$ and $b := U^T a$ lie both in $\mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^n$. By Proposition 3.8(b), it suffices to show

$$L_q((1 + w^T x)^2 (1 + b^T x)) \geq 0$$

where $q := p(Ux) \in \mathbb{R}[x]$. Here q and henceforth $r := q(x_1, x_2, 0, \ldots, 0) \in \mathbb{R}[x_1, x_2]$ are of course also real zero polynomials. By Lemma 3.26(a), it is enough to show

$$L_r((1 + w^T x)^2 (1 + b^T x)) \geq 0.$$

Now observe that $a \in C(p)$ implies $b \in C(q)$ and thus $(b_1, b_2) \in C(r)$. But $C(r) \subseteq S(r)$ by the already proven case $n = 2$. Hence $M_r(b_1, b_2) \succeq 0$ and we can conclude by Lemma 3.22. $\square$

3.4. **Relaxing linear programs.** We now come back to the most basic example of hyperbolic polynomials, namely products of linear polynomials non-vanishing at the origin. The rigidly convex sets they define are exactly the polyhedra containing the origin in their interior. The complexity behavior of optimization of linear functions over polyhedra is mainly governed by the number of linear inequalities they are defined by [MG]. For polyhedra with a huge number of facets, it is therefore reasonable to try to find reasonable outer approximations defined by a small linear matrix inequality. This fits of course into the above more general framework. However, in this special case, we get new interpretations, simplifications and extensions of the construction presented above:

First, we will be able to interpret the matrix coefficients A_i of the pencil from Definition 3.19 as moment matrices and localization matrices [Lau]. This might remind the reader of Lasserre's moment relaxations. But Lasserre's relaxation is a lift-and-project method where "moment matrices" are actually matrices filled with unknowns having the structure of moment matrices. Our method is not a lift-and-project method and the matrices are actual moment matrices filled with real numbers.

Second, the proofs for the case of linear programming will simplify dramatically. In particular, we will not need any version of the Helton-Vinnikov Theorem 2.7.

Third, we will not restrict to moments of degree three in this case but will go to arbitrarily high moments and thus present a hierarchy of relaxations for which we can prove finite convergence at level $d-1$ if d is the number of linear inequalities.

Proposition 3.36. Let $d \in \mathbb{N}_0$ and $a_1, \ldots, a_d \in \mathbb{R}^n$ and $p := \prod_{i=1}^d (1 + a_i^T x)$. Then

$$L_p(q) = \sum_{i=1}^{d} q(a_i)$$

for all $q \in \mathbb{R}[x]$, i.e., L_p is integration with respect to the sum of the Dirac measures in the points a_i.

Proof. In the case $d = 0$, we have $p = 1$ and thus $L_p = 0$. The case $d \geq 2$ reduces to the case $d = 1$ by means of Proposition 3.6(b). Hence we suppose now that $d = 1$ and write $a := a_1$. By linearity, it suffices to treat the case where q is a monomial. For the constant monomial $q = 1$, we have that $L_p(q) = \deg p = 1 = q(a)$. For the other monomials, we have to show

$$-\sum_{k=1}^{\infty} \frac{(a_1^T x)^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| = k}} \binom{|\alpha|}{\alpha} a_1^\alpha x^\alpha$$

in view of Definitions 3.4 and 3.2(b). But this follows from the multinomial theorem. □

The following generalizes Definition 3.19(b).

Definition 3.37. Let $p \in \mathbb{R}[x]$ with $p(0) \neq 0$ and $d \in \mathbb{N}_0$. Set $s := \binom{d+n}{d} = \binom{d+n}{n} = \#E$ where

$$E = \{x^\alpha \mid \alpha \in \mathbb{N}_0^n, |\alpha| \leq d\}$$

is the set of monomials in n variables of degree at most d. Fix an order on these monomials, i.e., write $E = \{m_1, \ldots, m_s\}$. Consider the symmetric matrices

$$A_0 := (L_p(m_i m_j))_{\substack{1 \leq i \leq s \\ 1 \leq j \leq s}} \qquad \text{and} \qquad A_k := (L_p(x_k m_i m_j))_{\substack{1 \leq i \leq s \\ 1 \leq j \leq s}}$$

for $k \in \{1, \ldots, n\}$. Then we call the linear matrix polynomial

$$M_p^{(d)} := A_0 + x_1 A_1 + \ldots + x_n A_n \in \mathbb{R}[x]^{s \times s}$$

the *d-th pencil associated to p* and

$$S^{(d)}(p) := \{a \in \mathbb{R}^n \mid M_p^{(d)}(a) \succeq 0\}$$

the *d-th spectrahedron associated to p*.

Lemma 3.38. Let $p \in \mathbb{R}[x]$ be a polynomial with $p(0) \neq 0$, $a \in \mathbb{R}^n$ and $d \in \mathbb{N}_0$. Let $m_1, \ldots, m_s$ be the pairwise distinct monomials in n variables of degree at most d in the order that has been fixed in Definition 3.37. Let $v \in \mathbb{R}^s$. Then

$$v^T M_p^{(d)}(a) v = L_p((v_1 m_1 + \ldots + v_s m_s)^2 (1 + a_1 x_1 + \ldots + a_n x_n)).$$

Lemma 3.39. Let $a_1, \ldots, a_d \in \mathbb{R}^n$ and $i \in \{1, \ldots, d\}$. Then there exists a polynomial $q \in \mathbb{R}[x] \setminus \{0\}$ with $\deg q < d$ such that $q(a_i) \neq 0$ and $q(a_j) = 0$ for all $j \in \{1, \ldots, d\} \setminus \{i\}$.

Proof. The polynomial $\prod_{j\in\{1,\dots,d\}\setminus\{i\}}(x^T(a_j-a_i))\neq 0$ cannot vanish on the whole of $\mathbb{R}^n$. So we can choose $v\in\mathbb{R}^n$ with $v^Ta_j\neq v^Ta_i$ for all $j\in\{1,\dots,d\}\setminus\{i\}$. Now set $q:=\prod_{j\in\{1,\dots,d\}\setminus\{i\}}(v^Ta_j-v^Tx)$ □

Theorem 3.40. Let $p\in\mathbb{R}[x]$ be a product of linear polynomials with $p(0)\neq 0$. Then the following hold:

(a) $C(p)\subseteq S^{(d)}(p)$ for each $d\in\mathbb{N}_0$ ("relaxation")
(b) $S^{(0)}(p)\supseteq S^{(1)}(p)\supseteq S^{(2)}(p)\supseteq S^{(3)}(p)\supseteq\dots$ ("hierarchy")
(c) If $d:=\deg p\geq 1$, then $C(p)=S^{(d-1)}(p)$ ("finite convergence")

Proof. WLOG $p(0)=1$. Write $p=\prod_{i=1}^d(1+a_i^Tx)$ with $a_1,\dots,a_d\in\mathbb{R}^n$. Then $d=\deg p$. For example by Proposition 2.12 (interpreting the product representation of p as a diagonal determinantal representation), we have

$$C(p)=\{a\in\mathbb{R}^n\mid 1+a_1^Tx\geq 0,\dots,1+a_d^Tx\geq 0\}.$$

(a) Let $a\in C(p)$. We have to show $M_{p,e}(a)\succeq 0$ for all $e\in\mathbb{N}_0$. By Lemma 3.38, this is equivalent to $L_p(q^2(1+a^Tx))\geq 0$ for all $q\in\mathbb{R}[x]$. This means by Proposition 3.36 that

$$\sum_{i=1}^d q(a_i)^2(1+a^Ta_i)\geq 0$$

for all $q\in\mathbb{R}[x]$. But even more is true: For each $i\in\{1,\dots,d\}$, $1+a^Ta_i=1+a_i^Ta\geq 0$ since $a\in C(p)$ and therefore $q(a_i)^2(1+a^Ta_i)\geq 0$.

(b) is clear from Definition 3.37.

(c) One inclusion has been proven already in (a). For the other one, let $a\in\mathbb{R}^n\setminus C(p)$. We show that $a\notin S^{(d-1)}(p)$. By Lemma 3.38 and Proposition 3.36, this means we have to show that there exists a polynomial $q\in\mathbb{R}[x]\setminus\{0\}$ with $\deg q\leq d-1$ such that

$$\sum_{j=1}^d q(a_j)^2(1+a_j^Ta)<0.$$

Choose $i\in\{1,\dots,d\}$ such that $1+a_i^Ta<0$. By Lemma 3.39, we can choose a polynomial $q\in\mathbb{R}[x]\setminus\{0\}$ with $\deg q<d$ such that $q(a_i)\neq 0$ and $q(a_j)=0$ for all $j\in\{1,\dots,d\}\setminus\{i\}$. Then

$$\sum_{j=1}^d q(a_j)^2(1+a_j^Ta)=q(a_i)^2(1+a_i^Ta)<0.$$

□

4. Tightening the relaxation

Let $p\in\mathbb{R}[x]$ be real zero polynomial. By Theorem 3.35, $S(p)$ is an outer spectrahedral approximation of $C(p)$, i.e., $C(p)\subseteq S(p)$. If p has high degree, then we cannot expect in general that this is a good approximation since $S(p)$ is defined by a very small linear matrix inequality. In this section, we analyze qualitatively a very simple idea of how to improve the spectrahedral outer approximation. The price we will have to pay is of course that we will need more linear matrix inequalities (of the same size however). Roughly, the idea is to "move the origin". More precisely, choose a point $a\in C(p)\setminus Z(p)$ that is different from the origin. By Theorem 2.14, the polynomial $p(x+a)$ is again a real zero polynomial and we have $C(p(x+a))+a = C(p)$. In general, we do however not have that $S(p(x+a))+a = S(p)$. This seems to a lacking theoretical property at first sight but turns out to be a fortunate fact that we can take advantage of. Namely, we have $C(p(x+a)) \subseteq S(p(x+a))$ by Theorem 3.35 and therefore $C(p) = C(p(x+a))+a \subseteq S(p(x+a))+a$ so that $S(p(x+a))+a$ is another outer spectrahedral relaxation of $C(p)$ that will in general be different from $S(p)$. Hence the intersection

$$S(p) \cap (S(p(x+a))+a)$$

will in general be an improved outer approximation of $C(p)$. It is defined by two linear matrix inequalities of size $n+1$ each which could of course be combined into a single one of size $2n+2$. Instead of choosing two points inside $C(p) \setminus Z(p)$, namely the origin and a, we could now more generally choose finitely many points $a_1, \dots, a_k \in C(p) \setminus Z(p)$ (the origin must not necessarily be among them) and consider the spectrahedron

$$\bigcap_{i=1}^{k} (S(p(x+a_i))+a_i) \supseteq C(p)$$

defined by a linear matrix inequality of size $k(n+1)$. In practice, it seems like $S(p(x+a))+a$ tightly approximates $C(p)$ in a neighborhood of $a \in C(p) \setminus Z(p)$. If this is right, then one would get a very tight outer approximation by choosing $a_1, \dots, a_k \in C(p) \setminus Z(p)$ in such way that each point in $C(p) \cap Z(p)$ is close to one of the a_i. This might of course have a very large price, namely that the number of points k might have to be very large.

Now we want to support the just presented view on how to might the approximation tighter. To this end, we prove a rather theoretical result that says that if we take *all* points of $C(p) \setminus Z(p)$ instead of just finitely many, then the corresponding intersection equals $C(p)$. This is Corollary 4.4 below. It does of course not imply that $C(p)$ is a spectrahedron since we deal now with an *infinite* intersection. In fact, we prove a more precise theorem that provides an evidence for our idea that it might be sufficient to choose the points close to the boundary of $C(p)$ provided each boundary point is close to one of the a_i. This is Theorem 4.3 where we intersect not over all points of $C(p) \setminus Z(p)$ but only over those lying outside of a fixed closed subset D of $C(p) \setminus Z(p)$. A good imagination is that $C(p)$ is a potato, $Z(p)$ ist skin and D the peeled potato (where the removed part inevitably is a bit more than the skin).

Our theoretical theorem actually works even for a certain polyhedral instead of spectrahedral outer approximation that we will now introduce. This is a poor man's version of the spectrahedron introduced in Definition 3.19(a). When no intersection comes into play, then we use actually just an affine half space or in exceptional cases the full space, namely the one that is defined by the linear inequality that corresponds to the top left entry of the pencil that defines the spectrahedron.

Definition 4.1. Let $p \in \mathbb{R}[[x]]$ be a power series with $p(0) \neq 0$ and $d \in \mathbb{N}_0$. We call

$$P_d(p) := \{a \in \mathbb{R}^n \mid L_{p,d}(1) + L_{p,d}(x_1)b_1 + \ldots + L_{p,d}(x_n)b_n \geq 0\}$$

the *polyhedron associated to p with respect to the virtual degree d*.

In Remark 6.23 below, we will give an interesting interpretation of this half-space for which we do not yet have the necessary notions.

Remark 4.2. (a) If $p \in \mathbb{R}[[x]]$ satisfies $\operatorname{trunc}_1 p = 1 + a_1x_1 + \ldots + a_nx_n$, then

$$P_d(p) = \{b \in \mathbb{R}^n \mid d + a_1b_1 + \ldots + a_nb_n \geq 0\}$$

by Example 3.5 and Definition 3.4(a).

(b) Let $p \in \mathbb{R}[[x]]$ with $p(0) \neq 0$, then $P_d(p)$ is either an affine half space or the full space depending on whether $\operatorname{trunc}_1 p$ is a constant polynomial or not.

(c) If $p \in \mathbb{R}[x]$ is a real zero polynomial of degree at most d, then we have

$$C(p) \subseteq S_d(p) \subseteq P_d(p)$$

where the first inclusion follows from Theorem 3.35 and the second is trivial.

(d) Let $p \in \mathbb{R}[x]$ is a real zero polynomial of degree at most d. The inclusion

$$C(p) \subseteq P_d(p)$$

is trivial contrary to the finer statement from (c). To show it, suppose WLOG $p(0) = 1$, $\deg p = d$ and write $\operatorname{trunc}_1 p = 1 + a_1x_1 + \ldots + a_nx_n$ with $a_1, \ldots, a_n \in \mathbb{R}$. We have to show that $d + a_1b_1 + \ldots + a_nb_n \geq 0$ for all $b \in C(p)$. Fixing $b \in C(p)$, write

$$(*) \qquad p(tb) = \prod_{i=1}^{e}(1 + d_it)$$

for some $e \in \{0, \ldots, d\}$ and $d_1, \ldots, d_e \in \mathbb{R}^\times$. By Definition 2.11, we have that $p(tb)$ has no roots in the interval $[0, 1)$. In other words, we have $d_i \geq -1$ for all $i \in \{1, \ldots, e\}$. Extracting the coefficient of t on both sides of $(*)$, we get $a_1b_1 + \ldots + a_nb_n = d_1 + \ldots + d_e$ and therefore

$$d + a_1b_1 + \ldots + a_nb_n = d + d_1 + \ldots + d_e \geq d - e \geq 0.$$

Theorem 4.3. Suppose $d \in \mathbb{N}_0$, $p \in \mathbb{R}[x]$ is a real zero polynomial of degree at most d and D a closed subset of $C(p) \setminus Z(p)$. Then

$$C(p) = \bigcap_{a \in C(p) \setminus (Z(p) \cup D)} S_d(p(x + a)) + a = \bigcap_{a \in C(p) \setminus (Z(p) \cup D)} P_d(p(x + a)) + a$$

where the empty intersection is interpreted as $\mathbb{R}^n$.

Proof. Recall that $p(x + a)$ is a real zero polynomial for each $a \in C(p) \setminus Z(p)$ by Theorem 2.14.

Both inclusions from left to right follow essentially from Theorem 3.35: For each $a \in C(p) \setminus Z(p)$, we have $C(p) = C(p(x + a)) + a$ and

$$C(p(x + a)) \subseteq S(p(x + a)) \subseteq S_d(p(x + a))$$

by Remark 4.2(c) so that $C(p) \subseteq S(p(x + a)) + a \subseteq P(p(x + a)) + a$.

It remains to show that

$$C(p) \supseteq \bigcap_{a \in C(p) \setminus (Z(p) \cup D)} P_d(p(x + a)) + a$$

We even show that for each half-line H emanating from the origin,

$$H \cap C(p) \supseteq H \cap \bigcap_{a \in H \cap (C(p) \setminus (Z(p) \cup D))} (P_d(p(x + a)) + a).$$

where the empty intersection stands of course again for $\mathbb{R}^n$. Using Proposition 3.8(a), one easily reduces to the case where H is the positive first axis

$$H = \mathbb{R}_{\geq 0} \times \{0\} \subseteq \mathbb{R}^n.$$

By Lemma 3.26(a), we can reduce to the case $n = 1$ since $p(x_1, 0, \dots, 0) \in \mathbb{R}[x_1]$ has again degree at most d. So suppose from now on that we have just one variable $x = x_1$. Then $H = \mathbb{R}_{\geq 0}$. WLOG $p(0) = 1$. Write

$$p = \prod_{i=1}^{d} (1 + a_i x)$$

with $a_1, \dots, a_d \in \mathbb{R}$. The roots of p are then the $-\frac{1}{a_1}, \dots, -\frac{1}{a_d}$. If none of these roots is positive, then $C(p) \cap H = \mathbb{R}_{\geq 0}$ and there is nothing to show. For now on we can therefore suppose that $d \geq 1$ and

$$r := -\frac{1}{a_1}$$

is the smallest positive root of p so that $H \cap C(p) = [0, r]$. In particular, $r > 0$ and $a_1 < 0$. Since D is closed and $r \notin D$, we can choose $\varepsilon > 0$ such that $r - \varepsilon > 0$ and $H \cap D \subseteq [0, r - \varepsilon]$. Then $(r - \varepsilon, r) \subseteq H \cap (C(p) \setminus (Z(p) \cup D))$. It therefore is enough to show that

$$(*) \qquad \bigcap_{a \in (r-\varepsilon, r)} P_d(p(x + a)) + a \subseteq (-\infty, r].$$

For $a \in \mathbb{R} \setminus Z(p) \supseteq (r - \varepsilon, r)$, the polynomial

$$(**) \qquad p_a := \frac{p(x + a)}{p(a)} = \prod_{i=1}^{d} \frac{1 + a_i(x + a)}{1 + a_i a} = \prod_{i=1}^{d} \left(1 + \frac{a_i}{1 + a_i a} x\right)$$

has constant coefficient 1. Proposition 3.36 hence implies that L_{p_a} is for each $a \in \mathbb{R} \setminus Z(p) \supseteq (r - \varepsilon, r)$ integration with respect to the sum of the Dirac measures in the points $\frac{a_1}{1+a_1 a}, \dots, \frac{a_d}{1+a_d a}$. Now suppose that b lies in the left hand side of $(*)$. Then we have

$$d + (b - a) \sum_{i=1}^{d} \frac{a_i}{1 + a_i a} = L_{p_a,d}(1) + (b - a) L_{p_a,d}(x) \geq 0$$

for all $a \in (r - \varepsilon, r)$. Now let a converge to r from below and consider what happens in $(*)$. We have that $1 + a_1 a$ converges to 0 from above. Hence the first term in the sum in $(**)$ converges to $-\infty$. The i-th term of the sum shows the same behavior in the case where $a_i = a_1$. All other terms of the sum converge to some real number. Hence the whole sum converges to $-\infty$. The term $b - a$ converges to $b - r$ from above. If $b - r$ wer positive, then the left hand side of $(*)$ would converge to $-\infty$ while being nonnegative all the time. Hence $b - r \leq 0$, i.e., b lies in the right hand side of $(*)$. $\square$

Corollary 4.4. Let $p \in \mathbb{R}[x]$ be a real zero polynomial. Then

$$C(p) = \bigcap_{a \in C(p) \setminus Z(p)} S(p(x + a)) + a.$$

5. EXACTNESS FOR QUADRATIC REAL ZERO POLYNOMIALS

It is quite trivial that $C(p) = S(p)$ for linear real zero polynomials $p \in \mathbb{R}[x]$. One way of seeing this is via Remark 4.2 which shows that $C(p) = P_1(p) = S(p)$ for all real zero polynomials $p \in \mathbb{R}[x]$ of degree 1. Since $S(p)$ depends only on the cubic part $\operatorname{trunc}_3 p$ of p by Remark 3.21, there seems to be no way that $C(p) = S(p)$ in general if the degree of of the real zero polynomial p is bigger than three. But worse than that, one lacks in general exactness also for cubic real zero polynomials as the reader will easily find. The next theorem shows that our relaxation at least is exact for quadratic real zero polynomials.

Theorem 5.1. Let p be a quadratic real zero polynomial. Then $C(p) = S(p)$.

Proof. From Theorem 3.35, we know already $C(p) \subseteq S(p)$. To show the reverse $S(p) \subseteq C(p)$, we show that it holds after intersecting with an arbitrary line through the origin. Instead of an arbitrary line, we can without loss of generality consider the first axis in $\mathbb{R}^n$ by Proposition 3.24. By Lemma 3.26(c), it is enough to show $S_{\deg p}(q) \subseteq C(q)$ for $q := p(x_1, 0, \ldots, 0) \in \mathbb{R}[x_1]$. If $\deg q = 0$, then $C(q) = \mathbb{R}$ and there is nothing to show. If $\deg q = 1$, say $q = 1 + ax_1$ with a non-zero $a \in \mathbb{R}$, then the bottom right entry of $M_{q,\deg p}$ is $a^2 + x_1 a^3$ by Proposition 3.36 and hence $S_{\deg p}(q) \subseteq \{b \in \mathbb{R} \mid a^2 + a^3 b \geq 0\} = \{b \in \mathbb{R} \mid 1 + ab \geq 0\}$. Finally, if $\deg q = 2$, then $S_{\deg p}(q) = S(q) = S^{(1)}(q) = C(q)$ by Theorem 3.40. $\square$

The preceding proof makes use of the Helton-Vinnikov Theorem 2.7 or Corollary 2.8 indirectly through Theorem 3.35. However, we need it only for quadratic polynomials. So we can use Example 2.9 instead and therefore our proof is self-contained.

The next corollary was explicitly mentioned by Netzer and Thom [NT, Corollary 5.4] but most likely was known before. Netzer and Thom use a hermitian linear matrix inequality of size $2^{\lfloor \frac{n}{2} \rfloor}$ to describe a rigidly convex set given by a quadratic polynomial in $\mathbb{R}^n$ whereas we need just a symmetric linear matrix inequality of size only $n+1$. While the proof of Netzer and Thom is still very interesting for other reasons, the result could actually also be proven by reducing it, via projective space, to the case of the unit ball (cf. [Går, Page 958]). The well-known description of the unit ball in $\mathbb{R}^n$ by a symmetric linear matrix inequality of size n (see for example [Kum1, Page 591]) shows that one can for $n \geq 2$ even get down to size n instead of $n+1$. If $n = 2^k + 1$ for some $k \in \mathbb{N}_0$, then Kummer shows that n is the minimal size of a real symmetric linear matrix inequality describing the unit ball. In general, he shows that $\frac{n}{2}$ is a lower bound [Kum1, Theorem 1].

Corollary 5.2. The rigidly convex set defined by a quadratic real zero polynomial is always a spectrahedron.

If $p \in \mathbb{R}[x]$ is a quadratic real zero polynomial, then it follows easily from this corollary and from Proposition 8.4 below that there exists $q \in \mathbb{R}[x]$, $d \in \mathbb{N}_0$ and symmetric matrices $A_1, \ldots, A_n \in \mathbb{R}^{d \times d}$ such that

$$pq = \det(I_d + x_1 A_1 + \ldots + x_n A_n)$$

and $C(p) \subseteq C(q)$. Our aim is now to prove this without using Proposition 8.4 by studying $\det(M_p)$ and see which cofactor q we get. We begin with a technical lemma that in particular gives a very nice determinantal representation of the real zero polynomial $1 - x_1^2 - \ldots - x_n^2$.

Lemma 5.3. Let $d_1, \ldots, d_n \in \mathbb{R}$. Consider the matrix

$$M := \begin{pmatrix} x_0 & -d_1x_1 & \ldots & -d_nx_n \\ -d_1x_1 & -d_1x_0 & & \\ \vdots & & \ddots & \\ -d_nx_n & & & -d_nx_0 \end{pmatrix} \in \mathbb{R}[x_0, x]^{(n+1)\times(n+1)}.$$

where the empty space is filled by zeros. Then

$$\det M = x_0^{n-1}(x_0^2 + d_1x_1^2 + \ldots + d_nx_n^2)\det(M(1,0)).$$

Proof. Setting $r := x_0^2 + d_1x_1^2 + \ldots + d_nx_n^2$, we show that $x_0 \det M = x_0^n r \det(M(1,0))$. Of course, $x_0 \det M$ is the determinant of the matrix N that arises from M by multiplying the first row with x_0. To compute the determinant of N, we subtracting x_i times its $(i+1)$-th row from its first row for each $i \in \{1, \ldots, n\}$. This results in an upper triangular matrix with diagonal entries $r, -d_1x_0, \ldots, -d_nx_0$. The determinant of N is the product of the diagonal entries, i.e., $x_0^n r(-d_1)\cdots(-d_n) = x_0^n r \det M(1,0)$. □

Using this lemma, we will now be able to compute the determinant of M_p for quadratic $p \in \mathbb{R}[x]$.

Theorem 5.4. For all quadratic $p \in \mathbb{R}[x]$ with $p(0) = 1$,

$$\det(M_p) = (\det(M_p(0)))\left(\frac{1 + \operatorname{trunc}_1 p}{2}\right)^{n-1} p.$$

Proof. For the constant polynomial $p = 1 \in \mathbb{R}[x]$, we have $L_p = 0$ by Definition 3.4 or Example 3.5 and hence $M_p = M_p(0) = 0 \in \mathbb{R}^{(n+1)\times(n+1)}$ by Definition 3.19. For a polynomial $p \in \mathbb{R}[x]$ of degree one, i.e., $p = b^Tx + 1$ for some $b \in \mathbb{R}^n$ with $b \neq 0$, Proposition 3.36 or Example 3.5 yields $M_p = (1 + b^Tx)M_p(0)$ and

$$M_p(0) = \begin{pmatrix} 1 & b^T \\ b^T & bb^T \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

has rank one and therefore vanishing determinant.

Hence it suffices to show the claim for all polynomials $p \in \mathbb{R}[x]$ of degree two. But for fixed degree polynomials p, both sides of the claimed equation depend continuously (in fact even polynomially) on the coefficients of p (the top left entry of $M_p(0)$ is then the constant degree and therefore makes no trouble). Hence it is enough to show the equations for all polynomials of the form

$$p = x^TAx + b^Tx + 1$$

for some symmetric matrix $A \in \mathbb{R}^{n\times n}$ and some vector $b \in \mathbb{R}^n$ such that $A \neq 0$ and $4A \neq bb^T$. Fix some p of this form. Set

$$\begin{aligned} q := p\left[-\frac{b}{2}\right] &= x^TAx + \left(1 - \frac{b^T}{2}x\right)b^Tx + \left(1 - \frac{b^T}{2}x\right)^2 \\ &= x^TAx + \left(-\frac{1}{2} + \frac{1}{4}\right)(b^Tx)^2 + 1 = x^T\left(A - \frac{1}{4}bb^T\right)x + 1. \end{aligned}$$

Choose an orthogonal matrix U such that

$$D := U^T\left(A - \frac{1}{4}bb^T\right)U$$

is diagonal and consider the polynomial

$$r := q(Ux) = x^T Dx + 1$$

of degree two. By Example 3.5, we have

$$M_r = \begin{pmatrix} 2 & -2x^T D \\ -2Dx & -2D \end{pmatrix} = M(1, x)$$

where

$$M := \begin{pmatrix} 2x_0 & -2x^T D \\ -2Dx & -2x_0 D \end{pmatrix} \in \mathbb{R}[x_0, x]^{(n+1)\times(n+1)}$$

By Lemma 5.3, we have

$$\det M = x_0^{n-1} r^* \det(M_r(0))$$

where $r^* \in \mathbb{R}[x_0, x]$ denotes the homogenization of r defined in Definition 3.9. By Remark 3.12, we have $p = q\left[\frac{b}{2}\right]$ so that Lemma 3.25 yields

$$\det(M_p) = \det\left(M_q + \frac{b^T x}{2} M_q(0)\right).$$

Rewriting the right hand side in view of $q = r(U^T x)$ by means of Lemma 3.23, we get

$$\begin{aligned} \det(M_p) &= \det\left(M_r(U^T x) + \frac{b^T x}{2} M_r(0)\right) = \det\left(M\left(1 + \frac{b^T x}{2}, U^T x\right)\right) \\ &= \left(1 + \frac{b^T x}{2}\right)^{n-1} (\det(M_r(0)))\, r^*\!\left(1 + \frac{b^T x}{2}, U^T x\right) \end{aligned}$$

Now we use $r^*(x_0, U^T x) = q^*$ to see that

$$\det(M_p) = (\det(M_r(0)))\left(\frac{1 + \operatorname{trunc}_1 p}{2}\right)^{n-1} q^*\left(1 + \frac{b^T x}{2}, x\right).$$

We result follows thus from

$$q^*\left(1 + \frac{b^T x}{2}, x\right) = q\left[\frac{b}{2}\right] = p.$$

$\square$

The following theorem follows easily from the previous one in the case where the matrix $M_p(0) \in \mathbb{R}^{(n+1)\times(n+1)}$ is invertible. In the general case, its proof is clearly inspired by the proof of the previous theorem.

Theorem 5.5. Let $p \in \mathbb{R}[x]$ be a quadratic real zero polynomial with $p(0) = 1$. Then there exist symmetric matrices $A_1, \ldots, A_n \in \mathbb{R}^{(n+1)\times(n+1)}$ such that

$$p\left(\frac{1 + \operatorname{trunc}_1 p}{2}\right)^{n-1} = \det(I_{n+1} + x_1 A_1 + \ldots + x_n A_n).$$

Proof. Write
$$p = x^T A x + b^T x + 1$$
where $A \in \mathbb{R}^{n\times n}$ is symmetric and $b \in \mathbb{R}^n$. Set
$$q := x^T \left(A - \frac{1}{4}bb^T\right) x + 1 \in \mathbb{R}[x].$$
Choose an orthogonal matrix U such that
$$D := U^T \left(A - \frac{1}{4}bb^T\right) U$$
is diagonal with diagonal entries
$$\underbrace{d_1, \dots, d_m}_{\substack{\text{all different}\\ \text{from zero}}}, \underbrace{0, \dots, 0}_{n-m \text{ zeros}}.$$
By Example 2.4, we have that each d_i is negative.

We first treat the special case where $m = 0$. In this case $A = \frac{1}{4}bb^T$ and thus $p = \left(1 + \frac{b^T x}{2}\right)^2$. Then we choose A_i as a diagonal matrix which arises from the identity matrix by replacing the first two diagonal entries by $\frac{1}{2}b_i$.

From now on, we consider the case $m > 0$. Then we consider the polynomial
$$r := q(Ux) = d_1 x_1^2 + \ldots + d_m x_m^2 + 1 \in \mathbb{R}[x_1, \dots, x_m]$$
of degree two and we set
$$M := \begin{pmatrix} x_0 & -d_1x_1 & \dots & -d_mx_m & & & \\ -d_1x_1 & -d_1x_0 & & & & & \\ \vdots & & \ddots & & & & \\ -d_mx_m & & & -d_mx_0 & & & \\ & & & & x_0 & & \\ & & & & & \ddots & \\ & & & & & & x_0 \end{pmatrix} \in \mathbb{R}[x_0, x]^{(n+1)\times(n+1)}$$
where the empty space is filled by zeros. Using Lemma 5.3, we see easily that
$$\det M = x_0^{n+1} r^* \det(M(1,0)).$$
Evaluating M in $(1 + \frac{b^T x}{2}, U^T x)$, we get another matrix
$$N \in \mathbb{R}[x]^{(n+1)\times(n+1)}$$
of the form $N = B_0 + x_1 B_1 + \ldots + x_n B_n$ with symmetric $B_0, \dots, B_n \in \mathbb{R}^{(n+1)\times(n+1)}$ such that
$$\begin{aligned}\det N &= \left(1 + \frac{b^T x}{2}\right)^{n+1} \left(q - 1 + \left(1 + \frac{b^T x}{2}\right)^2\right) \det(M(1,0)) \\ &= \left(1 + \frac{b^T x}{2}\right)^{n+1} p \det(M(1,0)) = c\left(\frac{1 + \operatorname{trunc}_1 p}{2}\right)^{n+1} p\end{aligned}$$
for some $c \in \mathbb{R}^\times$. Note that $B_0 = N(0) = M(1,0)$ is a diagonal matrix. Recalling that each d_i is negative, it is actually a diagonal matrix with only positive diagonal entries. Hence, it is easy to find an invertible diagonal matrix $D \in \mathbb{R}^{(n+1)\times(n+1)}$

such that $DB_0D = I_{n+1}$. Setting $A_i := DB_iD = D^2B_i$ for each $i \in \{0, \ldots, n\}$, we have $A_0 = I_{n+1}$ and

$$\det(A_0 + x_1A_1 + \ldots + x_nA_n) = c'\left(\frac{1 + \operatorname{trunc}_1 p}{2}\right)^k p$$

for some $c' \in \mathbb{R}$. Evaluating in 0, we see that $c' = 1$. □

As promised before Lemma 5.3, we have no found our cofactor

$$q := \left(\frac{1 + \operatorname{trunc}_1 p}{2}\right)^{n-1}$$

such that pq has the desired determinantal representation. Note that the required inclusion $C(p) \subseteq C(q)$ holds since

$$C(p) \subseteq P_2(p) = C\left(\frac{1 + \operatorname{trunc}_1 p}{2}\right) = C(q)$$

where the inclusion follows from Remark 4.2(c)(d).

Unfortunately, for real zero polynomials p of higher degree it is not true that one can always choose the cofactor q from Proposition 8.4 below to be a power of linear polynomial. This is discussed in detail in [AB, Section 7] where even other shapes for the cofactor are excluded. Indeed, from [AB, Theorem 7.3] (setting there $n := 8$ and $k := 2$) one can deduce the existence of a quartic real zero polynomial in 11 variables that it will never be of the shape $\det(I_N + x_1A_1 + \ldots + x_nA_n)$ with symmetric matrices $A \in \mathbb{R}^{N \times N}$ after being multiplied with a power of a linear polynomial. We are not aware of any cubic example like that. But at least the arguments in [Kum1, Example 12] show that

$$p := 10 - 3x_1^2 - 6x_2 - x_1^2x_2 - 3x_2^2 + x_2^3 - 3x_3^2 + x_2x_3^2 \in \mathbb{R}[x_1, x_2, x_3]$$

is a real zero polynomial such that that there is no linear polynomial $\ell \in \mathbb{R}[x_1, x_2, x_3]$ and $k \in \mathbb{N}$ such that $p\ell^k$ has a determinantal representation of the form

$$p\ell^k = \det(I_{3+k} + x_1A_1 + \ldots + x_nA_n)$$

with symmetric matrices $A_1, \ldots, A_n \in \mathbb{R}^{(3+k)\times(3+k)}$ (but they do not seem to exclude such a representation with matrices of size bigger than $3 + k$). Since p can easily be checked to be irreducible in $\mathbb{R}[x_1, x_2, x_3]$, [Kum1, Proposition 8] implies then together with Lemma 8.2 that there are a cubic $q \in \mathbb{R}[x_1, x_2, x_3]$ and symmetric matrices $A_1, A_2, A_3 \in \mathbb{R}^{6\times 6}$ such that $pq = \det(I_6 + x_1A_1 + x_2A_2 + x_3A_3)$.

6. Hyperbolicity cones and spectrahedral cones

Recall Definition 3.9 of homogeneous polynomials.

6.1. **Hyperbolic polynomials.**

The following is the homogeneous analog of Definition 2.1.

Definition 6.1. Let $p \in \mathbb{R}[x]$ be a homogeneous. We call a vector $e \in \mathbb{R}^n$ with $e \neq 0$ a *hyperbolicity direction* of p if for all $a \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$,

$$p(a - \lambda e) = 0 \implies \lambda \in \mathbb{R}.$$

In this case, we call p *hyperbolic with respect to* e or *hyperbolic in direction* e. We call p *hyperbolic* if it is hyperbolic with respect to some direction.

Remark 6.2. (a) If $p \in \mathbb{R}[x]$ is hyperbolic with respect to $e \in \mathbb{R}^n$, then $p(e) \neq 0$.
(b) If $p \in \mathbb{R}[x]$ is hyperbolic, then the number of variables n is of course greater or equal to 1 since $\mathbb{R}^0$ contains only the zero vector which can never be a hyperbolicity direction by definition.

The following is the analog of Proposition 2.3.

Proposition 6.3. Let $p \in \mathbb{R}[x]$ and $e \in \mathbb{R}^n$ with $e \neq 0$. Then p is hyperbolic in direction e if and only if for each $a \in \mathbb{R}^n$, the univariate polynomial

$$p(a - te) \in \mathbb{R}[t]$$

splits (i.e., is a product of non-zero linear polynomials) in $\mathbb{R}[t]$.

Proof. The "if" direction is easy and the "only if" direction follows from the fundamental theorem of algebra. $\square$

Definition 6.4. Let $p \in \mathbb{R}[x]$ be hyperbolic in direction e and let $a \in \mathbb{R}^n$.
(a) The zeros of $p(a - te) \in \mathbb{R}[t]$ are called the *eigenvalues* of a (with respect to p in direction e). When we speak of their *multiplicity*, we mean their multiplicity as roots of p.
(b) The weighted sum of the eigenvalues of a where the weights are the multiplicities is called the *trace* of a (with respect to p in direction e). We denote it by $\operatorname{tr}_{p,e}(a)$.

The following is the analog of Proposition 2.6.

Proposition 6.5. Let $A_1, \ldots, A_n \in \mathbb{C}^{d \times d}$ be hermitian matrices and $e \in \mathbb{R}^n$ with $e \neq 0$ such that $e_1 A_1 + \ldots + e_n A_n$ is definite. Then

$$p := \det(x_1 A_1 + \ldots + x_n A_n) \in \mathbb{R}[x]$$

is hyperbolic (of degree d) in direction e.

Proof. If $a \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ with $p(a - \lambda e) = 0$, then $\det(A - \lambda B) = 0$ where $A := a_1 A_1 + \ldots + a_n A_n \in \mathbb{C}^{d \times d}$ and $B := e_1 A_1 + \ldots + e_n A_n \in \mathbb{C}^{d \times d}$ is definite. We have to show $\lambda \in \mathbb{R}$. WLOG $\lambda \neq 0$. Then $\det(B + \frac{1}{\lambda}(-A)) = 0$ and thus $\frac{1}{\lambda} \in \mathbb{R}$ by Lemma 2.5. Hence $\lambda \in \mathbb{R}$. $\square$

6.2. **Hyperbolicity cones versus rigidly convex sets.** The following is the analog of Definition 2.11.

Definition 6.6. Let $p \in \mathbb{R}[x]$ be hyperbolic in direction e. Then we call

$$C(p, e) := \{a \in \mathbb{R}^n \mid \forall \lambda \in \mathbb{R} : (p(a - \lambda e) = 0 \implies \lambda \geq 0)\}$$

the *hyperbolicity cone* of p with respect to e.

A priori, it is not clear that hyperbolicity cones are cones. We will see this in Theorem 6.12 below but again it was already known to Gårding [Går, Theorem 2].

Proposition 6.7. Let $p \in \mathbb{R}[x]$. Then p is hyperbolic in direction of the first unit vector u of $\mathbb{R}^n$ if and only if its dehomogenization

$$q := p(1, x_2, \ldots, x_n) \in \mathbb{R}[x_2, \ldots, x_n]$$

is a real zero polynomial. In this case, we have for all $a \in \mathbb{R}^{n-1}$
(a) $(1, a) \in C(p, u) \iff a \in C(q)$ and
(b) $(0, a) \in C(p, u) \iff \forall \lambda \in \mathbb{R}_{\geq 0} : \lambda a \in C(q)$.

Proof. First, suppose that p is hyperbolic in direction of the first unit vector u. We show that q is a real zero polynomial. To this end, consider $a \in \mathbb{R}^{n-1}$ and $\lambda \in \mathbb{C}$ such that $q(\lambda a) = 0$. We have to show that $\lambda \in \mathbb{R}$. WLOG $\lambda \neq 0$. We show that $\mu := \frac{1}{\lambda} \in \mathbb{R}$. We have $p(u + \lambda(0, a)) = q(\lambda a) = 0$. By homogeneity, $p((0, a) + \mu u) = 0$. Since p is hyperbolic in direction u, we have indeed $\mu \in \mathbb{R}$.

Conversely, suppose that q is a real zero polynomial. To show that p is hyperbolic in direction u, we fix $a \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ such that $p(a - \lambda u) = 0$. We have to show that $\lambda \in \mathbb{R}$. If $\lambda = a_1$, then this is trivial. Hence suppose $\lambda \neq a_1$ and set $\mu := \frac{1}{a_1 - \lambda}$. It is enough to show that $\mu \in \mathbb{R}$. By homogeneity, we have $q(\mu a_2, \ldots, \mu a_n) = 0$ and thus $\mu \in \mathbb{R}$.

For the rest of the proof suppose that q is a real zero polynomial (in particular, $q(0) \neq 0$ as mentioned in Remark 2.2) and $a \in \mathbb{R}^{n-1}$.

(a) We have

$$
\begin{aligned}
(1, a) \in C(p, u) &\overset{\text{Definition 6.6}}{\iff} \forall \lambda \in \mathbb{R} : (p((1, a) - \lambda u) = 0 \implies \lambda \geq 0) \\
&\iff \forall \lambda \in \mathbb{R} : (p(1 - \lambda, a) = 0 \implies \lambda \geq 0) \\
&\iff \forall \lambda \in \mathbb{R}_{<0} : p(1 - \lambda, a) \neq 0 \\
&\iff \forall \lambda \in \mathbb{R}_{>0} : p(1 + \lambda, a) \neq 0 \\
&\overset{\text{homogeneity}}{\iff} \forall \lambda \in \mathbb{R}_{>0} : q\left(\frac{a}{1 + \lambda}\right) \neq 0 \\
&\iff \forall \lambda \in (0, 1) : q(\lambda a) \neq 0 \\
&\overset{q(0) \neq 0}{\iff} \forall \lambda \in [0, 1) : q(\lambda a) \neq 0 \\
&\overset{\text{Definition 2.10}}{\iff} a \in C(q).
\end{aligned}
$$

(b) We observe

$$
\begin{aligned}
(0, a) \in C(p, u) &\overset{\text{Definition 6.6}}{\iff} \forall \lambda \in \mathbb{R} : (p((0, a) - \lambda u) = 0 \implies \lambda \geq 0) \\
&\iff \forall \lambda \in \mathbb{R} : (p(-\lambda, a) = 0 \implies \lambda \geq 0) \\
&\iff \forall \lambda \in \mathbb{R}_{<0} : p(-\lambda, a) \neq 0 \\
&\iff \forall \lambda \in \mathbb{R}_{>0} : p(\lambda, a) \neq 0 \\
&\overset{\text{homogeneity}}{\iff} \forall \lambda \in \mathbb{R}_{>0} : q\left(\frac{a}{\lambda}\right) \neq 0 \\
&\iff \forall \lambda \in \mathbb{R}_{>0} : q(\lambda a) \neq 0 \\
&\overset{q(0) \neq 0}{\iff} \forall \mu \in \mathbb{R}_{\geq 0} : \forall \lambda \in [0, 1) : q(\lambda \mu a) \neq 0 \\
&\overset{\text{Definition 2.10}}{\iff} \forall \mu \in \mathbb{R}_{\geq 0} : \mu a \in C(q).
\end{aligned}
$$

□

6.3. **The homogeneous Helton-Vinnikov theorem.**

Remark 6.8. Let R be a ring. We can multiply finitely many matrices over this ring provided that for each factor (but the last one) its number of columns matches the number of rows of the next factor. Row and column vectors of elements can of course also be factors of such a product since they can be seen as matrices with a single row or column, respectively. Because of associativity of the matrix product, one does not have to specify parentheses in such a product of matrices over R. Now if M is an R-module, we can declare the product $AB \in M^{k\times m}$ of a matrix $A \in R^{k\times \ell}$ and a matrix $B \in M^{\ell\times m}$ in the obvious way. In this way, we can now also declare products of finitely many matrices exactly as above even if the last factor is not a matrix over R but over M. One has again the obvious associativity laws that allow to omit parentheses. We will use this in the proofs of Theorem 6.9 and Proposition 6.21 below for the case $R = \mathbb{R}[x]$ and $M = \mathbb{R}[x]^{d\times d}$. For example, we will write $x^T A$ for $x_1A_1 + \ldots + x_nA_n$ if A is the column vector with entries $A_1, \ldots, A_n \in \mathbb{C}^{d\times d}$.

The following is the homogeneous version of the Helton-Vinnikov Theorem 2.7.

Theorem 6.9 (Helton and Vinnikov)**.** If $p \in \mathbb{R}[x_1, x_2, x_3]$ is hyperbolic of degree d in direction $e \in \mathbb{R}^3$ such that $p(e) = 1$, then there exist symmetric $A_1, A_2, A_3 \in \mathbb{R}^{d\times d}$ such that $e_1A_1 + e_2A_2 + e_3A_3 = I_d$ and

$$p = \det(x_1A_1 + x_2A_2 + x_3A_3).$$

Proof. We reduce it to the Helton-Vinnikov theorem for real zero polynomials 2.7. Denote the first unit vector in $\mathbb{R}^3$ by u. Choose an orthogonal matrix $U \in \mathbb{R}^{3\times 3}$ such that $Ue = u$ and set $q := p(U^T x)$. Then $q \in \mathbb{R}[x_1, x_2, x_3]$ is also homogeneous of degree d but is hyperbolic in direction u. By Proposition 6.7, the polynomial $r := q(1, x_2, x_3) \in \mathbb{R}[x_2, x_3]$ is a real zero polynomial of some degree $d' \in \{0, \ldots, d\}$. By the Helton-Vinnikov theorem, we can choose symmetric $B_2', B_3' \in \mathbb{R}^{d'\times d'}$ such that $r = \det(I_{d'} + x_2B_2' + x_3B_3')$ (note that $r(0) = q(u) = q(Ue) = p(e) = 1$). Appending $d - d'$ zero columns and lines to B_2' and B_3', we get symmetric matrices $B_2, B_3 \in \mathbb{R}^{d\times d}$ such that $r = \det(I_d + x_2B_2 + x_3B_3)$. Setting $B_1 := I_d$, we see that $q = \det(x_1B_1 + x_2B_2 + x_3B_3) = \det(x^T B)$ where B designates the row vector with entries B_1, B_2, B_3 and we use the notation introduced in Remark 6.8. Hence $p = q(Ux) = \det((Ux)^T B) = \det(x^T U^T B) = \det(x^T A)$ where $A := U^T B$ is a row vector whose entries are symmetric matrices $A_1, A_2, A_3 \in \mathbb{R}^{d\times d}$. It remains to check that $e^T A = I_d$. Indeed, $e^T A = e^T U^T B = (Ue)^T B = u^T B = B_1 = I_d$. □

The following is the homogeneous version Corollary 6.10 which is a weaker version of the Helton-Vinnikov Theorem 6.9. It can be derived from Corollary 2.8 in exactly the same manner as we derived Theorem 6.9 from Theorem 2.7.

Corollary 6.10 (Helton and Vinnikov)**.** If $p \in \mathbb{R}[x_1, x_2, x_3]$ is hyperbolic of degree d in direction $e \in \mathbb{R}^3$ such that $p(e) = 1$, then there exist hermitian $A_1, A_2, A_3 \in \mathbb{C}^{d\times d}$ such that $e_1A_1 + e_2A_2 + e_3A_3 = I_d$ and

$$p = \det(I_d + x_1A_1 + x_2A_2 + x_3A_3).$$

6.4. **Basics on hyperbolicity cones.** The following is the analog of Proposition 2.12.

Proposition 6.11. Let $d \in \mathbb{N}_0$, $A_1, \ldots, A_n \in \mathbb{C}^{d\times d}$ be hermitian, $e \in \mathbb{R}^n$, $e \neq 0$

$$e_1A_1 + \ldots + e_nA_n \succ 0$$

and

$$p = \det(x_1A_1 + \ldots + x_nA_n).$$

Then

$$C(p,e) = \{a \in \mathbb{R}^n \mid a_1A_1 + \ldots + a_nA_n \succeq 0\}$$

and

$$C(p,e) \setminus Z(p) = \{a \in \mathbb{R}^n \mid a_1A_1 + \ldots + a_nA_n \succ 0\}.$$

Proof. The second statement follows easily from the first. To prove the first, set $A_0 := e_1A_1 + \ldots + e_nA_n$, let $a \in \mathbb{R}^n$ and set $B := a_1A_1 + \ldots + a_nA_n$. We have to show

$$(\forall \lambda \in \mathbb{R} : (\det(B - \lambda A_0) = 0 \implies \lambda \geq 0)) \iff B \succeq 0.$$

Since A_0 is positive definite, there exists a (unique) positive definite matrix $\sqrt{A_0}$ matrix whose square is A_0. Rewriting both the left and right hand side of our claim, it becomes

$$(\forall \lambda \in \mathbb{R} : (\det(C - \lambda I_d) = 0 \implies \lambda \geq 0)) \iff C \succeq 0$$

where $C := \sqrt{A_0}^{-1} B \sqrt{A_0}^{-1}$. This is clear. □

The following is the analog of Theorem 2.14.

Theorem 6.12 (Gårding)**.** Let $p \in \mathbb{R}[x]$ be hyperbolic in direction e and $a \in C(p,e) \setminus Z(p)$ with $a \neq 0$. Then a is also a hyperbolicity direction of p and $C(p,e) = C(p,a)$.

Proof. We suppose that e and a are linearly independent since otherwise all statements are trivial.

We first show that a is also a hyperbolicity direction of p. Now let $b \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ such that $p(b - \lambda a) = 0$. We have to show $\lambda \in \mathbb{R}$. The case where b is a linear combination of e and a ist again easy and we leave it to the reader. Hence we can now suppose that e, a and b are linearly independent. By an affine transformation, we can even suppose that these are the first three unit vectors in $\mathbb{R}^n$. Without loss of generality, we can thus assume that the number of variables is $n = 3$. Also WLOG $p(e) = 1$. By the Helton-Vinnikov Corollary 6.10, we can write

$$p = \det(x_1I_d + x_2A + x_3B)$$

with hermitian $A, B \in \mathbb{C}^{d \times d}$ where $d := \deg p$. The hypothesis $a \in C(p,e) \setminus Z(p)$ now translates into $A \succ 0$ by Proposition 6.11. From $\det(B - \lambda A) = 0$ and Lemma 2.5, we get $\lambda \in \mathbb{R}$.

To prove the second statement, fix $b \in \mathbb{R}^n$. We show that

$$(*) \qquad b \in C(p,e) \iff b \in C(p,a).$$

If b is a linear combination of e and a, this is a an exercise that we leave to the reader (make a case distinction according to the signs of the coefficients in this linear combination). From now on suppose that e, a and b are linearly independent. Suppose therefore that a and b are linearly independent. After an affine transformation, we can even assume that e, a and b are the first three unit vectors. Hence we can reduce to the case where the number of variables n equals 3. By the Helton-Vinnikov Corollary 6.10, we can choose hermitian matrices $A, B \in \mathbb{C}^{d \times d}$ such that

$$p = \det(x_1I_d + x_2A + x_3B)$$

so that

$$C(p,e) = \{c \in \mathbb{R}^3 \mid c_1I_d + c_2A + c_3B \succeq 0\}$$

by Proposition 6.11. Since $A \succ 0$ as mentioned above, Proposition 6.11 furthermore gives

$$C(p,a) = \{c \in \mathbb{R}^3 \mid c_1 I_d + c_2 A + c_3 B \succeq 0\}.$$

But now of course $C(p,e) = C(p,a)$. $\square$

Now we come to the analog of Theorem 2.15

Theorem 6.13 (Gårding)**.** Let $p \in \mathbb{R}[x]$ be hyperbolic in direction e. Then both $(C(p,e) \setminus Z(p)) \cup \{0\}$ and $C(p,e)$ are cones.

Proof. We start with $C := (C(p,e) \setminus Z(p)) \cup \{0\}$. We have to show $0 \in C$, $C + C \subseteq C$ and $\mathbb{R}_{\geq 0} C \subseteq C$. One checks immediately the first and the third property. For the second one, let $a, b \in C$. By Theorem 6.12, a and b are then hyperbolicity directions of p and we have $C(p,a) = C(p,e) = C(p,b)$. Since $a \in C(p,b)$, we have $p(a+b) = p(a-(-1)b) \neq 0$. It remains to show that $a + b \in C(p,e)$. Because of $C(p,e) = C(p,b)$, we can equivalently show that $a + b \in C(p,b)$. To this end, let $p(a + b - \lambda b) = 0$. We have to show that $\lambda \geq 0$. Because of $p(a - (\lambda - 1)b) = 0$ and $a \in C(p,b)$, we have even $\lambda - 1 \geq 0$.

To prove that $C(p,e)$ is also convex, we observe that

$$\begin{aligned}
C(p,e) &= \bigcap_{\varepsilon > 0} \{a \in \mathbb{R}^n \mid \forall \lambda \in \mathbb{R} : (p(a - \lambda e) = 0 \implies \lambda + \varepsilon > 0)\} \\
&= \bigcap_{\varepsilon > 0} \{a \in \mathbb{R}^n \mid \forall \lambda \in \mathbb{R} : (p(a - (\lambda - \varepsilon) e) = 0 \implies \lambda > 0)\} \\
&= \bigcap_{\varepsilon > 0} (\{a \in \mathbb{R}^n \mid \forall \lambda \in \mathbb{R} : (p(a - \lambda e) = 0 \implies \lambda > 0)\} - \varepsilon e) \\
&= \bigcap_{\varepsilon > 0} ((C(p,e) \setminus Z(p)) - \varepsilon e)
\end{aligned}$$

is an intersection of convex sets. $\square$

6.5. **Relaxing conic hyperbolic programs.**

Definition 6.14. Let $p \in \mathbb{R}[[x]]$ be a power series with $p(0) \neq 0$ and $d \in \mathbb{N}_0$. Consider the symmetric matrices $A_0, A_1, \ldots, A_n \in \mathbb{R}^{(n+1)\times(n+1)}$ from Definition 3.19. Then denote

$$M^*_{p,d} := x_0 A_0 + x_1 A_1 + \ldots + x_n A_n = x_0 M_{p,d}\left(\frac{x}{x_0}\right) \in \mathbb{R}[x]^{(n+1)\times(n+1)}.$$

The following is just a slight generalization of Lemma 3.22.

Lemma 6.15. Let $p \in \mathbb{R}[x]$ be a power series with $p(0) \neq 0$, $d \in \mathbb{N}_0$, $a = (a_0\, a_1\, \ldots\, a_n)^T \in \mathbb{R}^{n+1}$ and

$$v = (v_0\, v_1\, \ldots\, v_n)^T \in \mathbb{R}^{n+1}.$$

Then $v^T M^*_{p,d}(a) v = L_{p,d}((v_0 + v_1 x_1 + \ldots + v_n x_n)^2 (a_0 + a_1 x_1 + \ldots + a_n x_n))$.

Proof. Completely analogous to the proof of Lemma 3.22. $\square$

The following is a homogeneous version of Lemma 3.23(a).

Lemma 6.16. Suppose $U \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and consider the orthogonal matrix

$$\widetilde{U} := \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}.$$

Denote by $\widetilde{x}$ the column vector of variables $x_0, \dots, x_n$. If $p \in \mathbb{R}[[x]]$ is a power series with $p(0) \neq 0$ and $d \in \mathbb{N}_0$, then

$$M^*_{p(Ux),d} = \widetilde{U}^T M^*_{p,d}(\widetilde{U}\widetilde{x})\widetilde{U}.$$

Proof. This follows easily by homogenization from Lemma 3.23(a). □

Lemma 6.17. Let $e \in \mathbb{R}^n$ and $p \in \mathbb{R}[x]$ such that $e \neq 0$ and $p(e) \neq 0$. Let $d \in \mathbb{N}_0$. Denote the first unit vector in $\mathbb{R}^n$ by u. Let $U_1, U_2 \in \mathbb{R}^{n\times n}$ be orthogonal matrices such that $U_i e = \|e\| u$ for $i \in \{1,2\}$. Setting $q_i := p(U_i^T x) \in \mathbb{R}[x]$ and $r_i := q_i(1, x_2, \dots, x_n) \in \mathbb{R}[x_2, \dots, x_n]$, we have $r_i(0) \neq 0$ for $i \in \{1,2\}$ and

$$U_1^T M^*_{r_1,d}(U_1 x) U_1 = U_2^T M^*_{r_2,d}(U_2 x) U_2.$$

Proof. It is clear that $r_i(0) = q_i(e_i) = p(e) \neq 0$ for $i \in \{1,2\}$. The matrix $\widetilde{W} := U_1 U_2^T \in \mathbb{R}^{n\times n}$ is orthogonal and satisfies $\widetilde{W} u = u$. Hence $\widetilde{W}$ can be written in the form

$$\widetilde{W} = \begin{pmatrix} 1 & 0 \\ 0 & W \end{pmatrix} \in \mathbb{R}^{n\times n}$$

for some orthogonal matrix $W \in \mathbb{R}^{(n-1)\times(n-1)}$. We have $q_1(\widetilde{W}x) = q_2$ and therefore $r_1(Wy) = r_2$ where y is the column vector with entries $x_2, \dots, x_n$. By Lemma 6.16, we have

$$M^*_{r_2,d} = \widetilde{W}^T M^*_{r_1,d}(\widetilde{W}x)\widetilde{W}.$$

Replacing here x by $U_2 x$, we get

$$M^*_{r_2,d}(U_2 x) = U_2 U_1^T M^*_{r_1,d}(U_1 x) U_1 U_2^T$$

and thus

$$U_2^T M^*_{r_2,d}(U_2 x) U_2 = U_1^T M^*_{r_1,d}(U_1 x) U_1$$

as desired. □

The following is the analog of Definition 3.19.

Definition 6.18. Denote the first unit vector in $\mathbb{R}^n$ by u. Let $p \in \mathbb{R}[x]$ be hyperbolic in direction e of degree d and choose an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ such that $Ue = \|e\| u$. Consider the polynomial $q := p(U^T x)$ which obviously is hyperbolic in direction u and the polynomial $r := q(1, x_2, \dots, x_n) \in \mathbb{R}[x_2, \dots, x_n]$ which is a real zero polynomial by Proposition 6.7. Then the homogeneous linear matrix polynomial

$$M_{p,e} := U^T M^*_{r,d}(Ux) U \in \mathbb{R}[x]^{n\times n}$$

does not depend on the choice of U by Lemma 6.17. We call it the the *pencil associated to p with respect to e*. Moreover, we call the cone

$$S(p,e) := \{a \in \mathbb{R}^n \mid M_{p,e}(a) \succeq 0\} = \{a \in \mathbb{R}^n \mid M^*_{r,d}(Ua) \succeq 0\}$$

the *spectrahedral cone associated to p with respect to e*.

Remark 6.19. Let $p \in \mathbb{R}[x]$ be hyperbolic in direction e. Then it is obvious that λp is hyperbolic in direction μe and

$$S(p,e) = S(\lambda p, \mu e)$$

for all $\lambda, \mu \in \mathbb{R}$ with $\mu > 0$.

Proposition 6.20. Let $p \in \mathbb{R}[x]$ be hyperbolic in direction e.

(a) If p is non-constant, then the directional derivative

$$D_e p = \frac{d}{dt} p(x+te)|_{t=0}$$

of p in direction e is again hyperbolic in direction e.

(b) For all $a \in \mathbb{R}^n$, we have

$$e^T M_{p,e}(a) e = \|e\|^2 \operatorname{tr}_{p,e}(a).$$

(c) The map

$$\mathbb{R}^n \to \mathbb{R},\ a \mapsto \operatorname{tr}_{p,e}(a)$$

is linear.

Proof. (a) follows essentially from Definition 6.1 and from Rolle's theorem. To prove the other statements, we can suppose WLOG $\|e\| = 1$ and $p(e) = 1$ by Remark 6.19 and Definition 6.4. Set $d := \deg p$ and denote the first unit vector in $\mathbb{R}^n$ by u. Choose an orthogonal matrix $U \in \mathbb{R}^{n \times n}$ such that $Ue = u$ and consider $q := p(U^T x)$ which is hyperbolic in direction u and $r := q(1, x_2, \ldots, x_n) \in \mathbb{R}[x_2, \ldots, x_n]$ which is a real zero polynomial with $r(0) = q(u) = p(e) = 1$. Write $\operatorname{trunc}_1 r = 1 + b_2 x_2 + \ldots + b_n x_n$ with $b_2, \ldots, b_n \in \mathbb{R}$ and set $c := Ua \in \mathbb{R}^n$. By the Definitions 6.14, 3.19 and 6.18 as well as Example 3.5, we have

$$\begin{aligned} e^T M_{p,e}(a) e &= e^T U^T M^*_{r,d}(c) U e = u^T M^*_{r,d}(c) u \\ &= L_{r,d}(1) c_1 + L_{r,d}(X_2) c_2 + \ldots + L_{r,d}(X_n) c_n \\ &= d c_1 + b_2 c_2 + \ldots + b_n c_n. \end{aligned}$$

Now fix $a \in \mathbb{R}^n$ and write $p(a - te) = p(e) \prod_{i=1}^d (\lambda_i - t)$ with $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$. The coefficient of t^{d-1} in the univariate polynomial $f := q(Ua - tu) = p(a - te) \in \mathbb{R}[t]$ is $(-1)^{d-1} p(e) \operatorname{tr}_{p,e}(a) = (-1)^{d-1} \operatorname{tr}_{p,e}(a)$. Since the coefficients of the monomials

$$x_1^d, x_1^{d-1} x_2, \ldots, x_1^{d-1} x_n$$

in the polynomial q are $1, b_2, \ldots, x_n$, respectively, it is an easy exercise to see that the coefficient of t^{d-1} in f is also given by

$$(-1)^{d-1}(d c_1 + b_2 c_2 + \ldots + b_n c_n) = (-1)^d e^T M_{p,e}(a) e.$$

It follows that $e^T M_{p,e}(a) e = \operatorname{tr}_{p,e}(a)$. Since $a \in \mathbb{R}^n$ was arbitrary and $c = Ua$ depends of course linearly on a, we get also (c). $\square$

The following is a sharpening of Proposition 3.33:

Proposition 6.21. Suppose $d \in \mathbb{N}_0$ and $A_1, \ldots, A_n \in \mathbb{C}^{d \times d}$,

$$U := \{v_1 A_1 + \ldots + v_n A_n \mid v_1, \ldots, v_n \in \mathbb{R}\}$$

and (U, V) is an admissible couple (in particular, each A_i is hermitian). Set

$$p := \det(x_1 A_1 + \ldots + x_n A_n) \in \mathbb{R}[x]$$

and let $e \in \mathbb{R}^n \setminus \{0\}$ such that

$$e_1 A_1 + \ldots + e_n A_n = I_d.$$

(a) We have

$$C(p,e) = \{a \in \mathbb{R}^n \mid \forall M \in V : \operatorname{tr}(M^2(a_1A_1 + \ldots + a_nA_n)) \geq 0\} \qquad \text{and}$$
$$S(p,e) = \{a \in \mathbb{R}^n \mid \forall M \in U : \operatorname{tr}(M^2(a_1A_1 + \ldots + a_nA_n)) \geq 0\}.$$

(b) $C(p) \subseteq S(p,e)$

(c) If U is perfect, then $C(p) = S(p,e)$.

Proof. The first claim in (a) follows immediately from Proposition 6.11 together with Definition 3.29. To prove the second statement in (a), we choose an orthogonal matrix $W \in \mathbb{R}^{n\times n}$ such that $We = \|e\|u$ where u denotes the first unit vector in $\mathbb{R}^n$. Set $q := p(W^Tx)$ and consider the real zero polynomial $r := q(1, x_2, \ldots, x_n) \in \mathbb{R}[x_2, \ldots, x_n]$. According to Definition 6.18, we now have

$$S(p,e) = \{a \in \mathbb{R}^n \mid M^*_{r,d}(Wa) \succeq 0\}.$$

We now use a lot the notation explained in Remark 6.8. Write A for the column vector with entries $A_1, \ldots, A_n$. Then $B := WA$ is again a column vector whose entries are matrices $B_1, \ldots, B_n \in \mathbb{C}^{n\times n}$. Because of $A = W^TB$, we obviously have that

$$U = \{v^TA \mid v \in \mathbb{R}^n\} = \{v^TB \mid v \in \mathbb{R}^n\},$$

i.e., the A_i generate the same real vector space as the B_i. Substituting x by W^Tx in the equation $p = \det(x^TA)$ that defines p, we get

$$q = p(W^Tx) = \det((W^Tx)^TA) = \det(x^TWA) = \det(x^TB).$$

Observing that $B_1 = u^TB = u^TWA = (W^Tu)^TA = e^TA = I_n$, we see that

$$r = q(1, x_2, \ldots, x_n) = \det(I_d + x_2B_2 + \ldots + x_nB_n)$$

and hence

$$v^TM^*_{r,d}v = \operatorname{tr}((v^TB)^2(x^TB)).$$

for all $v \in \mathbb{R}^n$ due to Lemma 6.15. Substituting here Wa for x, we get

$$v^TM^*_{r,d}(Wa)v = \operatorname{tr}((v^TB)^2(Wa)^TB) = \operatorname{tr}((v^TB)^2a^TW^TB) = \operatorname{tr}((v^TB)^2a^TA)$$

for all $a, v \in \mathbb{R}^n$ and thus

$$M^*_{r,d}(Wa) \succeq 0 \iff \forall M \in U : \operatorname{tr}(M^2a^TA) \geq 0$$

for all $a \in \mathbb{R}^n$. □

We can now prove the homogeneous version of Theorem 3.35. For polynomials that have a hermitian determinantal representation like in Proposition 6.21, it follows immediately from that lemma. For other polynomials, we will again need the Helton-Vinnikov theorem, this time in its form of Corollary 6.10.

Theorem 6.22. Let $p \in \mathbb{R}[x]$ be hyperbolic in direction e. Then $C(p,e) \subseteq S(p,e)$.

Proof. For $n \leq 3$, the claim follows from Proposition 6.21(b) where we use the Helton-Vinnikov Corollary 6.10 for $n = 3$. Note that for $n \leq 2$, the A_i in Proposition 6.21 can obviously be chosen to be diagonal and we do not need Helton-Vinnikov.

We now suppose $n > 3$ and reduce it to the already proven case $n = 3$. Fix $a \in C(p,e)$ and $v \in \mathbb{R}^n$. We have to show

$$(*) \qquad v^T M_{p,e}(a) v \geq 0.$$

Denote again by u the first unit vector in $\mathbb{R}^n$ and choose an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ such that $Ue = u$, $w := Uv \in \mathbb{R}^3 \times \{0\} \subseteq \mathbb{R}^n$ and $b := Ua \in \mathbb{R}^3 \times \{0\} \subseteq \mathbb{R}^n$. Set $q := p(U^T x)$ and consider the real zero polynomial

$$r := q(1, x_2, \dots, x_n) \in \mathbb{R}[x_2, \dots, x_n].$$

By Definition 6.18, our Claim $(*)$ can now more concretely be formulated as

$$(**) \qquad w^T M^*_{r,d}(b) w \geq 0$$

where $d := \deg p$. By means of Lemma 6.15, the claim becomes

$$(***) \qquad L_{r,d}((w_1 + w_2 x_2 + w_3 x_3)^2 (b_1 + b_2 x_2 + b_3 x_3)) \geq 0$$

where we took into account that $w, b \in \mathbb{R}^3 \times \{0\} \subseteq \mathbb{R}^n$. Accordingly, we now consider the homogeneous polynomial

$$\tilde{q} := q(x_1, x_2, x_3, 0 \dots, 0) \in \mathbb{R}[x_1, x_2, x_3]$$

of degree d which is hyperbolic in direction u and the real zero polynomial

$$\tilde{r} := r(x_2, x_3, 0, \dots, 0) = \tilde{q}(1, x_2, x_3) \in \mathbb{R}[x_2, x_3].$$

By Lemma 3.26(a) applied on r and $\tilde{r}$, we can rewrite $(***)$ by

$$(****) \qquad L_{\tilde{r},d}((w_1 + w_2 x_2 + w_3 x_3)^2 (b_1 + b_2 x_2 + b_3 x_3)) \geq 0.$$

By Lemma 6.15 we are done if we can show $M^*_{\tilde{r},d}(b_1, b_2, b_3) \succeq 0$. By Definition 6.18 where one takes I_3 for the orthogonal matrix, this means that $(b_1, b_2, b_3) \in S(\tilde{q}, u)$. By the already treated case $n = 3$, it suffices to show that $(b_1, b_2, b_3) \in C(\tilde{q}, u)$. This is equivalent to $b \in C(q, u)$ which is in turn equivalent to our hypothesis $a \in C(p, e)$. □

Remark 6.23. (a) Let $p \in \mathbb{R}[x]$ be hyperbolic in direction e. Theorem 6.22 means that $v^T M_{p,e}(a) v \geq 0$ for all $a \in C(p,e)$ and $v \in \mathbb{R}^n$. For $v = e$, this means by Proposition 6.20(b) just that each element of the hyperbolicity cone $C(p,e)$ has nonnegative trace (with respect to p in direction e) which is clear by Definition 6.4 since it has even all eigenvalues nonnegative.

(b) Let $p \in \mathbb{R}[x]$ be a polynomial of degree at most d and denote by

$$q := x_0^d p\left(\frac{x}{x_0}\right) \in \mathbb{R}[x_0, x]$$

its degree d homogenization. Inspecting the proof of Proposition 6.20 (and using variables $x_0, \dots, x_n$ instead of $x_1, \dots, x_n$, the following enlightening interpretation of the polyhedron $P_d(p)$ defined in Definition 4.1 becomes now obvious: Its elements are those $b \in \mathbb{R}^n$ such that $(1, b) \in \mathbb{R}^{n+1}$ has nonnegative trace with respect to the hyperbolic polynomial q in direction of the first unit vector (confer Proposition 6.7). With this in mind, Remark 4.2(d) can now be read as an instance of the fact that an element has nonnegative trace if all its eigenvalues are nonnegative (with respect to an hyperbolic polynomial and an hyperbolicity direction). For the same reason, Item (a) of this remark can be seen as a generalization of Remark 4.2(d).

7. The determinant of the general symmetric matrix

In this section, we fix $d \in \mathbb{N}_0$, set $n := d + \frac{d^2-d}{2} = \frac{d(d+1)}{2}$,

$$\Lambda := \{(i,j) \in \{1,\dots,d\}^2 \mid i \le j\},$$

choose a bijection $\varrho\colon \Lambda \to \{1,\dots,n\}$ and set $x_{ij} := x_{\varrho(i,j)}$ for $(i,j) \in \Lambda$ so that $\mathbb{R}[x] = \mathbb{R}[x_{ij} \mid 1 \le i \le j \le n]$.

Given any vector a of length n, we write in the following

$$[a] := \begin{pmatrix} a_{\varrho(1,1)} & a_{\varrho(1,2)} & \cdots & a_{\varrho(1,d)} \\ a_{\varrho(1,2)} & a_{\varrho(2,2)} & \cdots & a_{\varrho(2,d)} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{\varrho(1,d)} & a_{\varrho(2,d)} & \cdots & a_{\varrho(d,d)} \end{pmatrix}$$

for the symmetric $d \times d$ matrix whose upper triangular part contains the entries of the "long vector" a (always with respect to the order prespecified by ϱ). Moreover, if $A = [a]$, then we write $a = \vec{A}$, i.e., $\vec{a}$ is a "long vector" that stores the entries in the upper triangular part of A. In the following we often identify $\mathbb{R}^n$ with the real vector space of symmetric $d \times d$ matrices by means of the vector space isomorphism $a \mapsto [a]$.

Definition 7.1. We call

$$X := [x] = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1d} \\ x_{12} & x_{22} & \dots & x_{2d} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ x_{1d} & x_{2d} & \dots & x_{dd} \end{pmatrix} \in \mathbb{R}[x]^{d\times d}$$

the *general symmetric matrix* of size d.

Proposition 7.2. The determinant $\det X$ of the general symmetric matrix X of size d is a homogeneous polynomial of degree d that is hyperbolic with respect to the identity matrix I_d. We have

$$C(\det X, I_d) = S(\det X, I_d) = \{A \in \mathbb{R}^{d\times d} \mid A \succeq 0\}.$$

Proof. It is clear that $\det X$ is homogeneous of degree d. For $(i,j) \in \Lambda$, denote by A_{ij} the unique symmetric matrix whose upper triangular part has zeros everywhere except for a one entry at position (i,j). Then

$$\det X = \det(x_1 A_1 + \ldots + x_n A_n)$$

and $A_1,\dots,A_n$ form a basis of the space of symmetric matrices which is perfect by Remark 3.31(a). Proposition 6.21(c) now says that $C(\det X, I_d) = S(\det X, I_d)$. Moreover, it follows easily from Proposition 6.11 that

$$C(\det X, I_d) = \{A \in \mathbb{R}^{d\times d} \mid A \succeq 0\}.$$

□

7.1. **Saunderson's representation of the derived cone.**

Theorem 7.3. Denote the standard unit vectors in $\mathbb{R}^n$ by $u_1, \dots, u_n$ and set $p := \det X \in \mathbb{R}[x]$. As in Definition 6.18, let U be an orthogonal matrix in $\mathbb{R}^{n\times n}$ such that $U\vec{I}_d = \|\vec{I}_d\|u_1$, consider $q := p(U^T x)$ and the real zero polynomial $r := q(1, x_2, \dots, x_n) \in \mathbb{R}[x_2, \dots, x_n]$. Consider the matrices $B_i := [U^T u_i] \in \mathbb{R}^{d\times d}$ for $i \in \{1, \dots, n\}$. Then the following hold:

(a)
$$M^*_{r,d}(Ux) = d\sqrt{d}\begin{pmatrix} \operatorname{tr}(B_1XB_1) & \dots & \operatorname{tr}(B_1XB_n) \\ \vdots & & \vdots \\ \operatorname{tr}(B_nXB_1) & \dots & \operatorname{tr}(B_nXB_n) \end{pmatrix} \in \mathbb{R}[x]^{n\times n}$$

(b) Suppose $d \geq 1$ so that $n \geq 1$ and consider the pencil N that arises from $M^*_{r,d}(Ux) \in \mathbb{R}[x]^{n\times n}$ by deleting the first row and the first column. The matrices $B_2, \dots, B_n$ form a basis of the vector space of symmetric trace zero matrices in $\mathbb{R}^{d\times d}$ so that N is by (a) essentially the pencil for which Saunderson [Sau, Theorem 2] has shown that the linear matrix inequality $N(x) \succeq 0$ defines the "first derivative relaxation of the cone of psd matrices", i.e.,
$$\{[a] \mid a \in \mathbb{R}^n, N(a) \succeq 0\} = C(D_{I_n} \det X, I_n)$$
where $D_{I_n} \det X$ is the directional derivative of $\det X$ in direction $\vec{I}_n$ which is hyperbolic in direction I_d by Proposition 6.20(a).

Proof. First note that

$$(*) \quad \sum_{i=1}^n (u_i^T Ux)B_i = \sum_{i=1}^n (u_i^T Ux)[U^T u_i] = \left[\sum_{i=1}^n (u_i^T Ux)U^T u_i\right]$$
$$= \left[U^T \sum_{i=1}^n (u_i^T Ux)u_i\right] = \left[U^T Ux\right] = [x] = X.$$

Substituting here x by $U^T x$, we get from this

$$(**) \qquad \sum_{i=1}^n x_i B_i = [U^T x].$$

(a) We have $p = \det X = \det([x_1u_1 + \dots + x_nu_n]) = \det([I_n x])$ where x is the column vector of variables x_i and thus
$$q = p(U^T x) = \det([U^T x]) \overset{(**)}{=} \det(x_1B_1 + \dots + x_nB_n).$$
Consequently, $r = \det(B_1 + x_2B_2 + \dots + x_nB_n)$. Because of $B_1 = [U^T u_1] = \frac{I_d}{\|\vec{I}_d\|} = \frac{I_d}{\sqrt{d}}$, we have $(\sqrt{d})^d r = \det(I_d + x_2\sqrt{d}B_2 + \dots + x_n\sqrt{d}B_n)$. Now Corollary 3.18 together with $\sqrt{d}B_1 = I_d$ implies that $L_{r,d}(1) = d\sqrt{d}\operatorname{tr}(B_1^3)$, $L_{r,d}(x_i) = d\sqrt{d}\operatorname{tr}(B_1^2B_i)$, $L_{r,d}(x_ix_j) = d\sqrt{d}\operatorname{tr}(B_1B_iB_j)$ and $L_{r,d}(x_ix_jx_k) = d\sqrt{d}\operatorname{tr}(B_iB_jB_k)$ for all $i, j, k \in \{1, \dots, n\}$. By Definition 3.19(a), the entry in row i and column j of $M^*_{r,d}(Ux)$ is thus
$$d\sqrt{d}\sum_{k=1}^n (u_k^T Ux)\operatorname{tr}(B_iB_kB_j) \overset{(*)}{=} d\sqrt{d}\operatorname{tr}(B_iXB_j)$$
for all $i, j \in \{1, \dots, d\}$.

(b) Since U^T is invertible, the vectors $\vec{B}_1, \dots, \vec{B}_{n-1}$ are linearly independent in $\mathbb{R}^n$. Since U^T is orthogonal, they are moreover orthogonal to $U^T\vec{u}_1 = \vec{I}_d$ with respect to the standard scalar product on $\mathbb{R}^n$. Summing up, $B_1, \dots, B_{n-1}$ are $n-1$ linearly independent symmetric matrices of trace zero and thus form a basis of the vector space of symmetric trace zero matrices in $\mathbb{R}^{d\times d}$ which has dimension $n-1$. □

8. THE GENERALIZED LAX CONJECTURE

The generalized Lax conjecture (GLC) has been stated by Helton and Vinnikov [HV, Section 6.1], see also [Ren, Page 63].

Conjecture 8.1 (generalized Lax conjecture, GLC)**.** Each rigidly convex set is a spectrahedron.

The following has been proven in [HV, Lemma 2.1] with heavy machinery from real algebraic geometry. We found an elementary approach which we present here.

Lemma 8.2. Let $p, q \in \mathbb{R}[x]$ be non-constant real zero polynomials such that $C(p) = C(q)$. Then p and q have a common non-constant factor in $\mathbb{R}[x]$.

Proof. For the duration of this proof, we call a line L through the origin in $\mathbb{R}^n$ *exceptional* if one of the polynomials p and q is constant on L. Then actually both are constant on L since if one of them, say p, is constant on L, then $L \subseteq C(p) = C(q)$ and so q, being a real zero polynomial, is also constant on L. If L is a non-exceptional line then p and q thus both vanish somewhere on it and because of $C(p) = C(q)$ they must even have a common root on L.

Now write $p = \sum_{i=0}^{d} p_i$ and $q = \sum_{j=0}^{e} q_j$ with $d := \deg p \geq 1$ and $e := \deg q \geq 1$ where $p_i \in \mathbb{R}[x]$ is homogeneous of degree i and $q_j \in \mathbb{R}[x]$ is homogeneous of degree j for all i and j. Consider now a kind of Sylvester matrix, namely

$$S := \begin{pmatrix}
p_d & p_{d-1} & \cdots\cdots & p_0 & & & \\
 & p_d & p_{d-1} & \cdots\cdots & p_0 & & \\
 & & \ddots & & & \ddots & \\
 & & & p_d & p_{d-1} & \cdots & p_0 \\
q_e & q_{e-1} & \cdots\cdots & q_0 & & & \\
 & q_e & q_{e-1} & \cdots\cdots & q_0 & & \\
 & & \ddots & & & \ddots & \\
 & & & q_e & q_{e-1} & \cdots & q_0
\end{pmatrix} \in \mathbb{R}[x]^{(d+e)\times(d+e)}$$

whose first e rows are consecutive shifts of $(p_d, \dots, p_0)$ and whose last d rows are consecutive shifts of $(q_e, \dots, q_0)$ where the empty space is filled up with zeros.

We claim that $S(a)$ is singular for all $a \in \mathbb{R}^n$. This is clear for $a = 0$ or if a spans an exceptional line since then the first column of $S(a)$ is zero. To prove the claim, it suffices to consider the case where a spans a non-exceptional line L. But on such a line p and q have a root in common so that there exists $\lambda \in \mathbb{R}$ with $p(\lambda a) = q(\lambda a) = 0$. Then $\sum_{i=0}^{d} p_i(a)\lambda^{i+k} = \lambda^k p(a) = 0$ and $q = \sum_{j=0}^{e} q_j(a)\lambda^{j+k} = \lambda^k q(a) = 0$ for all $k \in \mathbb{N}_0$ so that the transpose of the non-zero vector

$$\begin{pmatrix} \lambda^{d+e-1} & \lambda^{d+e-2} & \dots & \lambda^2 & \lambda & 1 \end{pmatrix}$$

lies in the kernel of $S(a)$. In particular, $S(a)$ is again singular.

By the claim just proven, we have $\det(S(a)) = 0$ for all $a \in \mathbb{R}^n$. This implies the polynomial identity $\det S = 0$. Hence S is singular as a matrix over the field $K(x) = K(x_1, \dots, x_n)$ of rational functions in the variables $x_1, \dots, x_n$. Over this field, there is thus a non-trivial linear dependance of its rows. Clearing denominators, this means that there exist a non-zero vector

$$v := \begin{pmatrix} f_1 & \dots & f_e & g_1 & \dots & g_d \end{pmatrix}^T \in \mathbb{R}[x]^{d+e}$$

such that $v^T S = 0$. Denoting by x_0 an additional variable, we multiply this from the right with the vector

$$w := \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{d+e-1} \end{pmatrix}^T \in \mathbb{R}[x_0]^{d+e}$$

and obtain

$$\underbrace{\sum_{i=1}^{e} f_i x_0^{i-1}}_{=:f \in \mathbb{R}[x_0, x]} p^* + \underbrace{\sum_{j=1}^{d} g_j x_0^{j-1}}_{=:g \in \mathbb{R}[x_0, x]} q^* = v^T S w = 0$$

where p^* and q^* are the homogenizations of p and q, respectively, as introduced in Definition 3.9. For every polynomial $h \in \mathbb{R}[x_0, x]$ we denote by $\deg_{x_0} h$ its degree with respect to x_0, i.e., the degree of h when it is seen as a polynomial in x_0 with coefficients from $K[x]$. We have $\deg_{x_0} g \leq d - 1 < d = \deg_{x_0} p^*$ so that p^* cannot divide g in $\mathbb{R}[x_0, x]$. Therefore, when we look at the prime factorization of

$$f p^* = -g q^*$$

in the factorial ring $\mathbb{R}[x_0, x]$, we find an irreducible factor r of p^* that divides q^* in $\mathbb{R}[x_0, x]$. Then $r(1, x)$ divides $q = q^*(1, x)$ in $\mathbb{R}[x]$. It remains to show that $r(1, x)$ cannot be constant. Indeed, the only way this could happen would be that x_0 divides r which is impossible since $r(0, x)$ divides $p^*(0, x) = p_d \neq 0$. $\square$

Lemma 8.3. Let $A_0, A_1, \dots, A_n \in \mathbb{R}^{d \times d}$ be symmetric such that the origin is an interior point of the spectrahedron

$$S := \{a \in \mathbb{R}^n \mid A_0 + a_1 A_1 + \ldots + a_n A_n \succeq 0\}.$$

Then there exists an invertible matrix $Q \in \mathbb{R}^{d \times d}$, $e \in \{0, \dots, d\}$ and symmetric matrices $B_1, \dots, B_n \in \mathbb{R}^{e \times e}$ such that

$$Q^T A_0 Q = \begin{pmatrix} I_e & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and}$$
$$Q^T A_i Q = \begin{pmatrix} B_i & 0 \\ 0 & 0 \end{pmatrix} \text{ for } i \in \{1, \dots, n\}.$$

Consequently,

$$S = \{a \in \mathbb{R}^n \mid I_e + a_1 B_1 + \ldots + a_n B_n \succeq 0\}.$$

Proof. Choose an orthogonal matrix $U \in \mathbb{R}^{d \times d}$ such that $U^T A_0 U$ is diagonal. Then choose a permutation matrix $P \in \mathbb{R}^{d \times d}$ such that the diagonal entries of (the diagonal) matrix $P^T U^T A_0 U P$ are $\lambda_1, \dots, \lambda_e$ followed by $d - e$ zeros. Since $0 \in S$, we have $A_0 \succeq 0$ so that $\lambda_1, \dots, \lambda_e$ are nonnegative. Let $D \in \mathbb{R}^{d \times d}$ be a diagonal matrix with diagonal entries $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_e}$ followed by $d-e$ arbitrary non-zero entries and set $Q := UPD^{-1} \in \mathbb{R}^{d\times d}$. Then Q is invertible and

$$Q^T A_0 Q = \begin{pmatrix} I_e & 0 \\ 0 & 0 \end{pmatrix}.$$

For each $i \in \{1, \dots, n\}$, $Q^T A_i Q$ is symmetric and can therefore be written as

$$Q^T A_i Q = \begin{pmatrix} B_i & C_i \\ C_i^T & D_i \end{pmatrix}$$

with $B_i \in \mathbb{R}^{e\times e}$, $C_i \in \mathbb{R}^{e\times(d-e)}$ and $D_i \in \mathbb{R}^{(d-e)\times(d-e)}$ such that B_i and D_i are symmetric. Since 0 is in the interior of S, there exists $\varepsilon > 0$ such that for all $\lambda \in \mathbb{R}$ with $-\varepsilon < \lambda < \varepsilon$ and all $i \in \{1, \dots, n\}$, we have $A_0 + \lambda A_i \succeq 0$ and thus $Q^T A_0 Q + \lambda Q^T A_i Q \succeq 0$. It follows that $D_i \succeq 0$ and $-D_i \succeq 0$ and thus $D_i = 0$ for all $i \in \{1, \dots, n\}$. Now it is an easy exercise to show that $C_i = 0$ for all $i \in \{1, \dots, n\}$. □

Proposition 8.4. Let $p \in \mathbb{R}[x]$ be a real zero polynomial. Then the following are equivalent:

(a) For each irreducible factor f of p in $\mathbb{R}[x]$, the rigidly convex set $C(f) \subseteq \mathbb{R}^n$ is a spectrahedron.

(b) There exist $q \in \mathbb{R}[x]$, $d \in \mathbb{N}_0$ and symmetric matrices $A_1, \dots, A_n \in \mathbb{R}^{d\times d}$ such that

$$pq = \det(I_d + x_1 A_1 + \ldots + x_n A_n)$$

and $C(p) \subseteq C(q)$.

In this case, $C(p)$ is a spectrahedron.

Proof. $C(p)$ is of course the intersection over the $C(f)$ where f runs over its irreducible factors. Using block diagonal matrices, one reduces therefore easily to the case where p is irreducible.

To show (a) $\Longrightarrow$ (b), we suppose that (a) holds. By Lemma 8.3, we find $d \in \mathbb{N}_0$ and symmetric matrices $A_1, \dots, A_n \in \mathbb{R}^{d\times d}$ such that

$$C(p) = \{a \in \mathbb{R}^n \mid I_d + a_1 A_1 + \ldots + a_n A_n\}.$$

Setting $r := \det(I_d + x_1 A_1 + \ldots + x_n A_n)$, we have $C(r) = C(p)$ by Proposition 2.12. By the irreducibility of p and Lemma 8.2, this implies that p divides r in $\mathbb{R}[x]$. Choose $q \in \mathbb{R}[x]$ such that $pq = r$. Since r is a real zero polynomial by Proposition 2.6, q is also a real zero polynomial. Moreover it is obvious that $C(p) \cap C(q) = C(r)$. Together with $C(r) = C(p)$, we obtain $C(p) \subseteq C(q)$.

To prove (b) $\Longrightarrow$ (a), let $q \in \mathbb{R}[x]$, $d \in \mathbb{N}_0$ and hermitian matrices $A_1, \dots, A_n \in \mathbb{C}^{d\times d}$ be given such that $pq = r := \det(I_d + x_1 A_1 + \ldots + x_n A_n)$ and $C(p) \subseteq C(q)$. Then $C(p) = C(p) \cap C(q) = C(r)$ is a spectrahedron by Proposition 2.6 □

Remark 8.5. If $C \subseteq \mathbb{R}^n$ is a cone, then $-C := \{-a \mid a \in C\}$ is again a cone and $C \cap -C$ is a subspace of $\mathbb{R}^n$ which is called the *lineality space* of C.

The lemma can also easily be deduced from [BGLS, Fact 2.9] proved by Gårding [Går, Theorem 3]. Here we give a very short proof based on the hermitian version of the result of Helton and Vinnikov.

Lemma 8.6. Let $p \in \mathbb{R}[x]$ be hyperbolic in direction of the first unit vector u of $\mathbb{R}^n$. Suppose that $m \in \{1, \dots, n\}$ and $C(p,u) \cap -C(p,u) = \{0\} \times \mathbb{R}^{n-m} \subseteq \mathbb{R}^n$. Then $p \in \mathbb{R}[x_1, \dots, x_m]$.

Proof. We have to show $p = p(x_1, \dots, x_m, 0, \dots, 0)$. It suffices to show $p(x_1, a, b) = p(x_1, a, 0)$ for all $a \in \mathbb{R}^{m-1}$ and $b \in \mathbb{R}^{n-m}$. Fix $a \in \mathbb{R}^{m-1}$ and $b \in \mathbb{R}^{n-m}$ and consider the polynomial $q := p(x_1, x_2a, x_3b) \in \mathbb{R}[x_1, x_2, x_3]$ of degree d which is obviously hyperbolic in direction of the first unit vector of $\mathbb{R}^3$. Because of the Helton-Vinnikov Corollary 6.9, we find hermitian matrices $A_2, A_3 \in \mathbb{C}^{d\times d}$ such that $q = \det(x_1 I_d + x_2 A_2 + x_3 A_3)$. Because of $(0,0,b) \in C(p,u) \cap -C(p,u)$, we have that all roots of the univariate polynomial $\det(A_3 - tI_d) = q(-t, 0, 1) = p(-t, 0, b) \in \mathbb{R}[t]$ are nonnegative and nonpositive and therefore zero. So all eigenvalues of the hermitian matrix A_3 are zero and therefore $A_3 = 0$. Consequently, $q = \det(x_1 I_d + x_2 A_2) \in \mathbb{R}[x_1, x_2]$. Hence $p(x_1, a, b) = q(x_1, 1, 1) = q(x_1, 1, 0) = p(x_1, a, 0)$ as desired. □

A stronger version of the following lemma is folklore and can be found for example in [Web, Theorem 2.5.1]. For convenience of the reader, we present the version that we need with the corresponding simplified proof.

Lemma 8.7. Let S be a closed unbounded convex set in $\mathbb{R}^n$ that contains the origin. Then S contains a ray, i.e., there exists $a \in \mathbb{R}^n \setminus \{0\}$ such that $\{\lambda a \mid \lambda \in \mathbb{R}_{\geq 0}\} \subseteq S$.

Proof. Choose a sequence $(a_i)_{i\in\mathbb{N}}$ in $S \setminus \{0\}$ such that $\lim_{i\to\infty} \|a_i\| = \infty$. Consider the sequence

$$\left(\frac{a_i}{\|a_i\|}\right)_{i\in\mathbb{N}}$$

of points on the unit sphere in $\mathbb{R}^n$. Since this sphere is compact, we may suppose that it converges to some point $a \in \mathbb{R}^n$ with $\|a\| = 1$. We claim that

$$\{\lambda a \mid \lambda \in \mathbb{R}_{\geq 0}\} \subseteq S.$$

To this purpose, we fix $\lambda \in \mathbb{R}_{\geq 0}$ and choose $k \in \mathbb{N}$ such that

$$\frac{\lambda}{\|a_i\|} \leq 1 \qquad \text{and thus} \qquad \frac{\lambda}{\|a_i\|} a_i \in S$$

for all $i \geq k$ because S is convex and contains the origin. It follows that

$$\lambda a = \lambda \lim_{i\to\infty} \frac{a_i}{\|a_i\|} = \lim_{i\to\infty} \frac{\lambda}{\|a_i\|} a_i \in S$$

since S is closed. □

Theorem 8.8 (equivalent formulations of GLC)**.** The following are equivalent:

(a) Each rigidly convex set is a spectrahedron, i.e., Conjecture 8.1 (GLC) holds.

(b) For each real zero polynomial $p \in \mathbb{R}[x]$, there exist a polynomial $q \in \mathbb{R}[x]$, some $d \in \mathbb{N}_0$ and symmetric matrices $A_1, \dots, A_n \in \mathbb{R}^{d\times d}$ such that

$$pq = \det(I_d + x_1 A_1 + \ldots + x_n A_n)$$

and $C(p) \subseteq C(q)$.

(c) Each hyperbolicity cone is spectrahedral.

(d) Each compact rigidly convex set is a spectrahedron.

Proof. (a) $\iff$ (b) follows directly from Proposition 8.4, (a) $\implies$ (d) is trivial. and (c) $\implies$ (a) follows immediately from Proposition 6.7.

It remains to show (d) $\implies$ (c). Suppose (d) holds and let $p \in \mathbb{R}[x]$ be hyperbolic in direction e. WLOG we suppose that p has positive degree as otherwise its hyperbolicity cone $C(p,e)$ is all of $\mathbb{R}^n$. By Remark 8.5, the lineality space

$$L := C(p,e) \cap -C(p,e) \subseteq \mathbb{R}^n$$

of the hyperbolicity cone $C(p,e)$ is a linear subspace of $\mathbb{R}^n$. It consists of all elements of $\mathbb{R}^n$ all of whose eigenvalues (with respect to p in direction e) in the sense of Definition 6.4(a) are zero. Since $C(p,e)$ consists of those elements all of whose eigenvalues are nonnegative, we see that

$$L = C(p,e) \cap H = -C(p,e) \cap H.$$

Thus L is of course contained in

$$H := \{a \in \mathbb{R}^n \mid \operatorname{tr}_{p,e}(a) = 0\} \subseteq \mathbb{R}^n$$

by Definition 6.4(b) which is a hyperplane by Proposition 6.20(c) with

$$e \notin H \supseteq L$$

due to the positive degree of p. In particular, $\dim L = n - m$ for some $m \in \{1, \dots, n\}$.

Claim 1. We can reduce to the case where

- e is the first unit vector of $\mathbb{R}^n$,
- $H = \{0\} \times \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$ and
- $L = \{0\} \times \mathbb{R}^{n-m} \subseteq \mathbb{R}^n$.

Justification. Choose an invertible matrix $A \in \mathbb{R}^{n \times n}$ whose first column is e, whose remaining columns span H and whose last $n - m$ columns span L, i.e., A maps

- the first unit vector u of $\mathbb{R}^n$ to e,
- the subspace $H' := \{0\} \times \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$ onto H and
- the subspace $L' := \{0\} \times \mathbb{R}^{n-m} \subseteq \mathbb{R}^n$ onto L.

Setting $q := p(Ax) \in \mathbb{R}[x]$, we have for each $a \in A$ the univariate polynomial identity $q(a - tu) = p(Aa - te) \in \mathbb{R}[t]$ which shows that p is hyperbolic in direction u with

$$C(p,e) = \{Aa \mid a \in C(q,u)\}$$

and $\operatorname{tr}_{q,u}(a) = \operatorname{tr}_{p,e}(Aa)$ for each $a \in A$. It follows that $H' = \{a \in \mathbb{R}^n \mid \operatorname{tr}_{q,u}(a) = 0\}$ and $L' = \{A^{-1}a \mid a \in L\} = C(q,u) \cap -C(q,u)$. This proves Claim 1.

Claim 2. We can further reduce to the case where

- e is the first unit vector of $\mathbb{R}^n$,
- $H = \{0\} \times \mathbb{R}^{n-1} \subseteq \mathbb{R}^n$ and
- $L = \{0\} \subseteq \mathbb{R}^n$.

Justification. Suppose we are already in the situation described in Claim 1. By Lemma 8.6, we then have that

$$p \in \mathbb{R}[x_1, \dots, x_m].$$

Viewed as a polynomial in the variables $x_1, \dots, x_m$, p is clearly again hyperbolic with respect to the first unit vector u of $\mathbb{R}^m$ and we have obviously

$$C(p,e) = C(p,u) \times \mathbb{R}^{n-m}$$

as well as $\operatorname{tr}_{p,e}(a,0) = \operatorname{tr}_{p,u}(a)$ for all $a \in \mathbb{R}^m$. It is therefore enough to show that the hyperbolicity cone $C(p,u) \subseteq \mathbb{R}^m$ is spectrahedral. Finally, we have that

$$\begin{aligned} H' &:= \{0\} \times \mathbb{R}^{m-1} = \{a \in \mathbb{R}^m \mid (a,0) \in H\} = \{a \in \mathbb{R}^m \mid \operatorname{tr}_{p,e}(a,0) = 0\} \\ &= \{a \in \mathbb{R}^m \mid \operatorname{tr}_{p,u}(a) = 0\} \end{aligned}$$

and $L' := C(p,u) \cap -C(p,u) = \{a \in \mathbb{R}^m \mid (a,0) \in L\} = \{0\}$. This proves Claim 2.

Claim 3. Suppose we are in the situation of Claim 2 and consider

$$q := p(1, x_2, \dots, x_n) \in \mathbb{R}[x_2, \dots, x_n].$$

Then q is a real zero polynomial and its associated rigidly convex set

$$C(q) = \{(a_2, \dots, a_n) \in \mathbb{R}^{n-1} \mid (1, a_2, \dots, a_n) \in C(p,e)\} \subseteq \mathbb{R}^{n-1}$$

is compact.

Justification. By Proposition 6.7(a), we only need to show that $C(q)$ is compact. Certainly, it is closed since, for example, it is an intersection of closed half-spaces by Theorem 4.3. By Lemma 8.7, it is enough to show that $C(q)$ does not contain a ray. By Proposition 6.7(b) this is equivalent to showing that $H \cap C(p,e) = \{0\}$ which is true by Claim 2 since $L = H \cap C(p,e)$. This proves Claim 3.

Claim 4. Suppose we are in the situation of Claim 2. Then

$$C(p,e) = \{0\} \cup \left\{ a \in \mathbb{R}^n \mid a_1 > 0, \frac{(a_2, \dots, a_n)}{a_1} \in C(q) \right\}.$$

Justification. The inclusion from right to left follows easily from Claim 3 and the fact that $C(p,e)$ is a cone by Theorem 6.13. For the other inclusion, let $a \in C(p,e) \setminus \{0\}$. By Claim 3, it suffices to show that $a_1 > 0$. Writing $a = a_1 e + b$ with $b \in H$, we see that $0 \leq \operatorname{tr}_{p,e}(a) = a_1(\deg p) + 0$. Since p has positive degree, it follows that $a_1 \geq 0$. Moreover, if we had $a_1 = 0$, it would follow that $\operatorname{tr}_{p,e}(a) = 0$ and hence $a \in C(p,e) \cap H = L = \{0\}$ by Claim 2 which contradicts $a \neq 0$. This proves Claim 4.

Now we can finally **conclude the proof** of (d) $\implies$ (c). The rigidly convex set $C(q) \subseteq \mathbb{R}^{n-1}$ is compact by Claim 3 is thus a spectrahedron by hypothesis (d). Accordingly, we can choose some $e \in \mathbb{N}_0$ and symmetric matrices $A_1, \dots, A_n \in \mathbb{R}^{e \times e}$ such that

$$C(q) = \{(a_2, \dots, a_n) \in \mathbb{R}^{n-1} \mid A_1 + a_2 A_2 + \ldots + a_n A_n \succeq 0\}.$$

We claim that

$$C(p,e) = \{a \in \mathbb{R}^n \mid a_1 \geq 0,\, a_1 A_1 + a_2 A_2 + \ldots + a_n A_n \succeq 0\}$$

so that the hyperbolicity cone $C(p,e)$ can be defined by a linear matrix inequality of size $e+1$ (in block diagonal form with a block of size 1 and a block of size e). The inclusion from left to right is immediately by the corresponding inclusion from Claim 4. The other inclusion follows from the other inclusion in Claim 4 if we can exclude that there exists $(a_2, \dots, a_n) \in \mathbb{R}^{n-1} \setminus \{0\}$ with $a_2 A_2 + \ldots + a_n A_n \succeq 0$. But this follows from the boundedness of $C(q)$ proved in Claim 3. $\square$

8.1. **The real zero amalgamation conjecture.**

Conjecture 8.9 (real zero amalgamation conjecture, RZAC)**.** Let $\ell, m, n \in \mathbb{N}_0$ and consider $\ell+m+n$ variables coming in three blocks $x = (x_1, \dots, x_\ell)$, $y = (y_1, \dots, y_m)$ and $z = (z_1, \dots, z_n)$. Let $d \in \mathbb{N}_0$ and suppose that $p \in \mathbb{R}[x, y]$ and $q \in \mathbb{R}[x, z]$ are real zero polynomials of degree at most d with

$$p(x, 0) = q(x, 0).$$

Then there exist a real zero polynomial $r \in \mathbb{R}[x, y, z]$ of degree at most d such that

$$r(x, y, 0) = p \qquad \text{and} \qquad r(x, 0, z) = q.$$

In the situation of Conjecture 8.9 (RZAC), we call $x_1, \dots, x_\ell$ the *shared variables* and r the *amalgamation polynomial*.

Remark 8.10. Conjecture 8.9 (RZAC) is of course trivially true in the case where p and q have determinantal representation

$$\begin{aligned} p &= \det(I_d + x_1 A_1 + \ldots + x_\ell A_\ell + y_1 B_1 + \ldots + y_m B_m) \qquad \text{and} \\ q &= \det(I_d + x_1 A_1 + \ldots + x_\ell A_\ell + z_1 C_1 + \ldots + z_n C_n) \end{aligned}$$

with hermitian matrices $A_i, B_j, C_k \in \mathbb{C}^{d \times d}$ since then

$$r := \det(I_d + x_1 A_1 + \ldots + x_\ell A_\ell + y_1 B_1 + \ldots + y_m B_m + z_1 C_1 + \ldots + z_n C_n)$$

is a real zero polynomial of degree at most d by Proposition 2.6.

The following easy lemma will be a key element in our proof of the amalgamation conjecture for $\ell = 0$ which we give in Theorem 8.15(a) below. In the search for a proof of the general amalgamation conjecture it could be useful to generalize this lemma appropriately to more than two variables but the obvious generalizations do not work as one easily sees (cf. the "crucial" argument in the proof of Lemma 8.11 below).

Lemma 8.11. Consider the polynomial ring $\mathbb{R}[x, y]$ in two single variables x and y. For all $d \in \mathbb{N}_0$ and $k, \ell \in \{0, \dots, d\}$, we have

$$\sum_{\substack{i,j \in \mathbb{N}_0 \\ i+j=d}} \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} (x^k y^\ell) = \begin{cases} 0 & \text{if } k+\ell < d, \\ \frac{k!l!}{(k+\ell-d)!}(x+y)^{k+\ell-d} & \text{if } d \le k+\ell. \end{cases}$$

Proof. The case $k+\ell < d$ is trivial. Therefore we suppose now $d \le k+\ell$. Then the left hand side equals

$$\sum_{\substack{i,j \in \mathbb{N}_0 \\ i+j=d \\ i \le k,\ j \le \ell}} k(k-1)\cdots(k-i+1)\,\ell(\ell-1)\cdots(\ell-j+1)(x^{k-i}y^{\ell-j}).$$

Changing the indices of summation (i instead of $k-i$ and j instead of $\ell-j$), we can rewrite this as

$$\sum_{\substack{i,j \in \mathbb{N}_0 \\ k-i+\ell-j=d \\ i \le k,\ j \le \ell}} k(k-1)\cdots(i+1)\,\ell(\ell-1)\cdots(j+1)(x^i y^j).$$

It is now crucial that the condition $i \leq k$ can here be omitted since it is implied by $i = k + \ell - j - d \leq k + d - d = k$. Analogously, the condition $j \leq \ell$ can also be omitted. Hence the expression in question equals

$$\sum_{\substack{i,j \in \mathbb{N}_0 \\ i+j=k+\ell-d}} \frac{k!}{i!} \frac{\ell!}{j!} x^i y^j = \frac{k!\ell!}{(k+\ell-d)!} \sum_{\substack{i,j \in \mathbb{N}_0 \\ i+j=k+\ell-d}} \frac{(k+\ell-d)!}{i!j!} x^i y^j$$

where the last sum is the binomial expansion of $(x+y)^{k+\ell-d}$. □

To formulate and prove the next lemma, we need some elements of the theory of multivariate real stable polynomials [BB1, BB2, Wag] (see also [Pem, Section 5]) which we will introduce for the convenience of the reader.

Definition 8.12. We call a real polynomial in one or several variables *stable* if if there is no complex zero all of whose components have positive imaginary part.

Remark 8.13. The following are obvious:

(a) Any product of stable polynomials is again stable.
(b) A univariate real polynomial is stable if and only if it is real-rooted, i.e., all of its complex zeros are real.

The following lemma is a special case of both [BB2, Theorem 3.4] and [BB2, Lemma 6.1]. However, we include here a proof since variants of it might be useful for proving Conjecture 8.9 (RZAC).

Lemma 8.14. Let V denote the real vector space of polynomials in two variables x and y which have at most degree d with respect to each of the variables x and y (i.e., V is the span of the monomials $x^k y^\ell$ with $0 \leq k, \ell \leq d$). Then the linear operator

$$T \colon V \to V,\ p \mapsto \sum_{\substack{i,j \in \mathbb{N}_0 \\ i+j=d}} \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} p$$

preserves stability [BB1, Page 542], i.e., for each stable $p \in V$ the image $T(p)$ is again stable unless it is the zero polynomial.

Proof. We use the characterization of linear stability preservers due to Borcea and Brändén [BB2, Theorem 3.1] (see also [Wag, Theorem 5.2] and [Lea, Theorem 1.1]). Namely, it suffices to prove that the *symbol* of T [BB1, Page 544]

$$\sum_{i=0}^{d} \sum_{j=0}^{d} \binom{d}{i} \binom{d}{j} T(x^i y^j) v^{d-i} w^{d-j} \in \mathbb{R}[v, w, x, y]$$

is stable where v and w are two additional variables. By Lemma 8.11, this symbol equals

$$\begin{aligned}
&\sum_{\substack{0\le i\le d\\0\le j\le d\\d\le i+j}}\binom{d}{i}\binom{d}{j}\frac{i!\,j!}{(i+j-d)!}(x+y)^{i+j-d}v^{d-i}w^{d-j}\\
=&\sum_{s=0}^{d}\sum_{i=s}^{d}\binom{d}{i}\binom{d}{s+d-i}\frac{i!\,(s+d-i)!}{s!}(x+y)^s v^{d-i}w^{i-s}\\
=&\sum_{s=0}^{d}\sum_{t=0}^{d-s}\binom{d}{t+s}\binom{d}{d-t}\frac{(t+s)!\,(d-t)!}{s!}(x+y)^s v^{d-s-t}w^t\\
=&\sum_{s=0}^{d}\frac{(x+y)^s}{s!}\sum_{t=0}^{d-s}\frac{(d!)^2}{(d-s-t)!\,t!}v^{d-s-t}w^t\\
=&d!\sum_{s=0}^{d}\frac{d!}{s!\,(d-s)!}(x+y)^s\sum_{t=0}^{d-s}\frac{(d-s)!}{(d-s-t)!\,t!}v^{d-s-t}w^t\\
=&d!\sum_{s=0}^{d}\binom{d}{s}(x+y)^s(v+w)^{d-s}=d!(v+w+x+y)^d
\end{aligned}$$

which is stable by Remark 8.13(a) since $v+w+x+y\in\mathbb{R}[v,w,x,y]$ is clearly stable. □

We are now able to prove three special cases of our amalgamation conjecture where we use for each case another non-trivial ingredient, namely the theory of stability preservers, the Helton-Vinnikov theorem and positive semidefinite matrix completion.

Theorem 8.15. Conjecture 8.9 (RZAC) holds true in each of the following cases:

(a) $\ell=0$, i.e., if there are no shared variables,
(b) $\ell=m=n=1$, i.e., if each block of variables consists just of a single variable,
(c) $d=2$, i.e., for quadratic polynomials.

Proof. (a) As announced, we will use the theory of stability preservers. Consider the *degree d homogenizations* with respect to x_0

$$\tilde p := x_0^d\, p\left(\frac{y}{x_0}\right)\in\mathbb{R}[x_0,y]\quad\text{and}\quad \tilde q := x_0^d\, q\left(\frac{z}{x_0}\right)\in\mathbb{R}[x_0,z].$$

By Proposition 6.7, both $\tilde p$ and $\tilde q$ are hyperbolic in direction of the first unit vector in $\mathbb{R}^{1+m}$ and $\mathbb{R}^{1+n}$, respectively. This means that for each $b\in\mathbb{R}^m$ and $c\in\mathbb{R}^n$, the univariate polynomials $p(x_0,b),q(x_0,c)\in\mathbb{R}[x_0]$ of degree d are real-rooted and therefore stable by Remark 8.13(b). We introduce now an additional variable $\hat x_0$ which one should think of as a copy of x_0.

Fix for the moment $b\in\mathbb{R}^m$ and $c\in\mathbb{R}^n$. Then the two polynomials $p(x_0,b),q(\hat x_0,c)\in\mathbb{R}[x_0,\hat x_0]$ are stable so that their product $p(x_0,b)q(\hat x_0,c)\in\mathbb{R}[x_0,\hat x_0]$ is stable by Remark 8.13(a). Hence the polynomial

$$r_{b,c}:=\sum_{i+j=d}\frac{\partial^i}{\partial x_0^i}\frac{\partial^j}{\partial \hat x_0^j}p(x_0,b)q(\hat x_0,c)\in\mathbb{R}[x_0,\hat x_0]$$

is stable by Lemma 8.14. By Lemma 8.11, we have actually $r_{b,c} \in \mathbb{R}[x_0 + \hat{x}_0]$, i.e., $r_{b,c}$ is a polynomial in $x_0 + \hat{x}_0$. In particular, $r_{b,c}(x_0, 0) = r_{b,c}(0, x_0) \in \mathbb{R}[x_0]$ is stable and thus real-rooted for all $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

Considering the homogeneous degree d polynomial

$$\tilde{r} := \sum_{i+j=d} \frac{\partial^i}{\partial x_0^i} \frac{\partial^j}{\partial \hat{x}_0^j} \tilde{p}\, \tilde{q}(\hat{x}_0, z) \in \mathbb{R}[x_0, \hat{x}_0, y, z],$$

we see again from Lemma 8.11 that $\tilde{r} \in \mathbb{R}[x_0 + \hat{x}_0, y, z]$ and thus $\tilde{r}(x_0, 0, y, z) = \tilde{r}(0, x_0, y, z) \in \mathbb{R}[x_0, y, z]$. From the preceding, we know that

$$\tilde{r}(x_0, 0, y, z) = \tilde{r}(0, x_0, y, z)$$

is hyperbolic in direction of the first unit vector of $\mathbb{R}^{1+m+n}$ so that by Proposition 6.7, its dehomogenization $r := \tilde{r}(1, 0, y, z) = \tilde{r}(0, 1, y, z) \in \mathbb{R}[y, z]$ is a real zero polynomial.

Finally, we show that $r(y, 0) = d!q(0)p$ and $r(0, z) = d!p(0)q$ so that

$$\frac{1}{d!p(0)} r = \frac{1}{d!q(0)} r$$

is an amalgamation polynomial that we have searched for. By symmetry, it suffices to shows the first claim, i.e., $r(y, 0) = d!q(0)p$. To this end, observe that substituting 0 for z commutes with taking the partial derivatives in the formula defining $\tilde{r}$. Together with $\tilde{q}(\hat{x}_0, 0) = q(0)\hat{x}_0^d$ this shows that when one sets x and z to zero in this formula, then only term of the sum on the right hand side that survives is the one corresponding to $i = 0$ and $j = d$. This terms then equals $d!q(0)\tilde{p}$. Setting x_0 to 1 yields the claim.

(b) WLOG $p(0) = q(0) = 1$. By the Corollary 2.8 to the Helton-Vinnikov theorem 2.7, we can choose hermitian matrices $A, A', B, C \in \mathbb{C}^{d \times d}$ such that

$$p = \det(I_d + xA + yB) \qquad \text{and} \qquad q = \det(I_d + xA' + zC).$$

By conjugating each of A and B with a suitable unitary matrix, we can without loss of generality assume that A is diagonal. Conjugating it once more with a suitable permutation matrix, we can moreover suppose that the (real) diagonal entries of A are weakly increasing. In the same way, we may assume that also A' is a diagonal matrix with weakly increasing diagonal. The diagonal entries of A can now be reconstructed from the polynomial $\det(I_d + xD)$ by looking at its degree, its roots and the multiplicities of its roots. The analogous statement holds for for A' and D'. Because of the polynomial identity $\det(I_d + xA) = p(x, 0) = q(x, 0) = \det(I_d + xA')$, we thus get $A = A'$. Now $r := \det(I_d + xA + yB + zC)$ is an amalgamation polynomial just like in Remark 8.10.

(c) WLOG $p(0) = q(0) = 1$. We will use the theory of positive semidefinite matrix completion from [GJSW]. By Example 2.4, there are

- symmetric matrices $A \in \mathbb{R}^{\ell \times \ell}$, $B \in \mathbb{R}^{m \times m}$ and $C \in \mathbb{R}^{n \times n}$,
- matrices $E \in \mathbb{R}^{\ell \times m}$ and $F \in \mathbb{R}^{\ell \times n}$ and
- vectors $a \in \mathbb{R}^{\ell}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$

such that

$$p = \begin{pmatrix} x^T & y^T \end{pmatrix} \begin{pmatrix} A & E \\ E^T & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a^T & b^T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 1 \qquad \text{and}$$

$$q = \begin{pmatrix} x^T & z^T \end{pmatrix} \begin{pmatrix} A & F \\ F^T & C \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} a^T & c^T \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + 1$$

where the both "discriminants"

$$P := \begin{pmatrix} aa^T - 4A & ab^T - 4E \\ ba^T - 4E^T & bb^T - 4B \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a^T & b^T \end{pmatrix} - 4 \begin{pmatrix} A & E \\ E^T & B \end{pmatrix} \in \mathbb{R}^{(\ell+m)\times(\ell+m)} \qquad \text{and}$$

$$Q := \begin{pmatrix} aa^T - 4A & ac^T - 4F \\ ca^T - 4F^T & cc^T - 4C \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} a^T & c^T \end{pmatrix} - 4 \begin{pmatrix} A & F \\ F^T & C \end{pmatrix} \in \mathbb{R}^{(\ell+n)\times(\ell+n)}$$

are positive semidefinite. The task is to find a matrix $G \in \mathbb{R}^{m\times n}$ such that the quadratic polynomial

$$r := \begin{pmatrix} x^T & y^T & z^T \end{pmatrix} \begin{pmatrix} A & E & G \\ E^T & B & F \\ G^T & F^T & C \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a^T & b^T & c^T \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 1$$

is a real zero polynomial, i.e., the "discriminant" of r

$$\begin{pmatrix} aa^T - 4A & ab^T - 4E & ac^T - 4G \\ ba^T - 4E^T & bb^T - 4B & bc^T - 4F \\ ca^T - 4G^T & cb^T - 4F^T & cc^T - 4C \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} a^T & b^T & c^T \end{pmatrix} - 4 \begin{pmatrix} A & E & G \\ E^T & B & F \\ G^T & F^T & C \end{pmatrix}$$

is positive semidefinite. Since a and c are now fixed vectors, this amounts to fill the blocks marked by a question mark in the "partial matrix"

$$\begin{pmatrix} aa^T - 4A & ab^T - 4E & ? \\ ba^T - 4E^T & bb^T - 4B & bc^T - 4F \\ ? & cb^T - 4F^T & cc^T - 4C \end{pmatrix}$$

by real numbers so that one obtains a positive semidefinite matrix of size $\ell + m + n$. This is a positive semidefinite matrix completion problem. The undirected graph G with loops whose edges correspond to the known entries is

$$G = \{1, \ldots, \ell + m\}^2 \cup \{m + 1, \ldots, \ell + m + n\}^2$$

is obtained by glueing together a complete graph on $\ell + m$ vertices with a complete graph on $m + n$ vertices along a complete graph on m vertices. It is an easy exercise to show that this graph is *chordal* in the sense of [GJSW], i.e., each cycle consisting of at least four pairwise distinct nodes in this graph has a chord. By [GJSW, Theorem 7] it follows that our matrix completion problem can be solved since the principal submatrices corresponding to cliques of the graph in the partial matrix are all positive semidefinite. Indeed, each such submatrix is a principal submatrix of the discriminants P and Q of p and q which are both positive semidefinite. □

Remark 8.16. We hope that our proof of Theorem 8.15(a) leads to ideas of how to generalize it to the case $\ell \geq 1$. The amalgamating polynomial

$$\frac{1}{d!p(0)} r = \frac{1}{d!q(0)} r$$

in its proof can obviously be written as

$$\sum_{i=0}^{d} \left(\frac{\partial^i}{\partial x_0^i} \tilde{p} \right)\bigg|_{x_0=1} \left(\frac{\partial^{d-i}}{\partial x_0^{d-i}} \tilde{q} \right)\bigg|_{x_0=0}.$$

In the special case $\ell = 0$ of Remark 8.10, i.e., if

$$p = \det(I_d + y_1 B_1 + \ldots + y_m B_m) \qquad \text{and}$$
$$q = \det(I_d + z_1 C_1 + \ldots + z_n C_n)$$

for some hermitian matrices $B_1, \ldots, B_m, C_1, \ldots, C_n \in \mathbb{C}^{d \times d}$, it follows easily from [MSS, Theorem 1.2] that this amalgamating polynomial equals

$$\int \det(I_d + y_1 B_1 + \ldots + y_m B_m + U^*(z_1 C_1 + \ldots + z_n C_n)U) dU$$

where the integral is a vector-valued integral (inside the finite-dimensional space of polynomials in $\mathbb{R}[y, z]$ of degree at most k) with respect to the Haar measure on the compact group of unitary matrices of size d (see also [Mar]). Our proof of Conjecture 8.9 (RZAC) for $\ell = 0$ is thus just a "multivariate variant" of the "univariate convolution" studied already by Walsh [Wal] (see also [RS, Section 5.3]) and it is closely related to both Remark 8.15 and the proof of Part (b) if Theorem 8.15. In the case $\ell \geq 1$, we feel that it could be useful to introduce copies $\hat{x}_1, \ldots, \hat{x}_\ell$ of the common variables $x_1, \ldots, x_\ell$ and treat them similarly as the "homogenizing variable" x_0 and its copy $\hat{x}_0$. Applying for each x_i and $\hat{x}_i$ the analogous differential operator that we have applied for x_0 and $\hat{x}_0$ seems a good idea except that reduces the degree way too much. We tried to fix these degree issues by amalgamating more than two polynomials at a time (which would be even better) but ran into problems with the version of Lemma 8.11 that we would need for that.

8.2. **Wrapping rigidly convex sets into spectrahedra and tying them with a cord.**

Lemma 8.17. Suppose that Conjecture 8.9 (RZAC) holds. Let $p \in \mathbb{R}[x_1, x_2, y_1, \ldots, y_m] = \mathbb{R}[x, y]$ be a real zero polynomial of degree $d \geq 2$. Set

$$n := \left(\frac{d(d+1)}{2} - 3 \right) \in \mathbb{N}_0.$$

Then there exists a real zero polynomial $q \in \mathbb{R}[x, y, z_1, \ldots, z_n] = \mathbb{R}[x, y, z]$ of degree d such that $q(x, y, 0) = p$,

$$C(p) \subseteq \{(a, b) \in \mathbb{R}^{2+m} \mid (a, b, 0) \in S(q)\} \qquad \text{and}$$
$$\{a \in \mathbb{R}^2 \mid (a, 0) \in C(p)\} = \{a \in \mathbb{R}^2 \mid (a, 0, 0) \in S(q)\}.$$

In particular, $C(p)$ is contained in a spectrahedron that agrees with $C(p)$ on the plane spanned by the first two unit vectors.

Proof. WLOG $p(0) = 1$. By the Helton-Vinnikov theorem 2.7, we find symmetric $A_1, A_2 \in \mathbb{R}^{d \times d}$ such that $p(x, 0) = \det(I_d + x_1 A_1 + x_2 A_2)$.

We now claim that we can find real symmetric matrices $B_1, \ldots, B_n \in \mathbb{R}^{d \times d}$ such that they span together with I_d, A_1 and A_2 a subspace of the space of real symmetric matrices of size d that is perfect in the sense of Definition 3.29. To this end, we distinguish three different cases. In each of these cases, we will use one of the subspaces that are perfect according to 3.31(a).

If A_1 and A_2 are scalar multiples of the identity matrix then we can for example set $B_i := 0$ for $i \in \{1, \dots, n\}$ and get the perfect subspace generated by I_d.

If the span of I_d, A_1 and A_2 is two-dimensional, then we can jointly diagonalize A_1 and A_2 by conjugating them with a suitable orthogonal matrix and therefore can assume them to be diagonal. Then the co-dimension of the span of I_d, A_1 and A_2 inside the space of real diagonal matrices is $d - 2 = (d + 1) - 3$ which is at most n because of $d \geq 2$. Hence we find diagonal matrices $B_1, \dots, B_n \in \mathbb{R}^{d \times d}$ such that the span of $I_d, A_1, A_2, B_1, \dots, B_n$ is the space of all diagonal matrices which is again perfect.

The last case is where I_d, A_1 and A_2 are linearly independent. Then we complete them to a basis $I_d, A_1, A_2, B_1, \dots, B_n$ of the perfect space of real symmetric matrices of size d.

The claim is now proven and we choose $B_1, \dots, B_n \in \mathbb{R}^{d \times d}$ according to it. By Conjecture 8.9 (RZAC), $p \in \mathbb{R}[x, y]$ and

$$r := \det(I_d + x_1 A_1 + x_2 A_2 + z_1 B_1 + \ldots + z_n B_n) \in \mathbb{R}[x, z]$$

can be amalgamated into a real zero polynomial $q \in \mathbb{R}[x, y, z]$ such that $q(x, y, 0) = p$ and $q(x, 0, z) = r$. By Theorem 3.35 applied to q, we have $C(q) \subseteq S(q)$. Together with Lemma 3.26(b), this yields our first statement

$$C(p) = \{(a, b) \in \mathbb{R}^{2+m} \mid (a, b, 0) \in C(q)\} \subseteq \{(a, b) \in \mathbb{R}^{2+m} \mid (a, b, 0) \in S(q)\}.$$

We have $C(r) = S(r)$ by Proposition 3.33(c). Together with Lemma 3.26, we get

$$\begin{aligned} \{a \in \mathbb{R}^2 \mid (a, 0) \in C(p)\} &= \{a \in \mathbb{R}^2 \mid (a, 0, 0) \in C(q)\} \\ &= \{a \in \mathbb{R}^2 \mid (a, 0) \in C(r)\} = \{a \in \mathbb{R}^2 \mid (a, 0) \in S(r)\} \\ &\supseteq \{a \in \mathbb{R}^2 \mid (a, 0, 0) \in S(q)\} \supseteq \{a \in \mathbb{R}^2 \mid (a, 0) \in C(p)\} \end{aligned}$$

where the last inclusion follows from the already proven part of the lemma. □

Lemma 8.18. Suppose that Conjecture 8.9 (RZAC) holds. Let $p \in \mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_\ell]$ be a real zero polynomial of degree $d \geq 2$. Set

$$m := \left(\frac{d(d+1)}{2} - 3 \right) \in \mathbb{N}_0.$$

Let U be a two-dimensional subspace of $\mathbb{R}^\ell$. Then there exists a real zero polynomial $q \in \mathbb{R}[x_1, \dots, x_\ell, y_1, \dots, y_m] = \mathbb{R}[x, y]$ of degree d such that $q(x, 0) = p$ and the spectrahedron $S := \{a \in \mathbb{R}^\ell \mid (a, 0) \in S(q)\} \subseteq \mathbb{R}^\ell$ satisfies $C(p) \subseteq S$ and $U \cap C(p) = U \cap S$.

Proof. Choose an orthogonal matrix $Q \in \mathbb{R}^{\ell \times \ell}$ such that $\{Qx \mid x \in U\}$ equals the span of the first two unit vectors U' in $\mathbb{R}^\ell$. Applying Lemma 8.17 to the real zero polynomial $p' := p(Qx)$, we obtain a real zero polynomial $q' \in \mathbb{R}[x, y]$ of degree d with $q'(x, 0) = p'$ such that the spectrahedron $S' := \{a \in \mathbb{R}^\ell \mid (a, 0) \in S(q')\}$ satisfies $C(p') \subseteq S'$ and $U' \cap C(p') = U' \cap S'$. Consider now the orthogonal matrix

$$Q' := \begin{pmatrix} Q & 0 \\ 0 & I_m \end{pmatrix} \in \mathbb{R}^{(\ell+m) \times (\ell+m)}$$

and the real zero polynomial

$$q := q'\left(Q'^T \begin{pmatrix} x \\ y \end{pmatrix} \right) = q'(Q^T x, y) \in \mathbb{R}[x, y]$$

of degree d. Then $q(x,0) = q'(Q^{-1}x,0) = p'(Q^{-1}x) = p$. Moreover, we have $C(p) = \{Q^T a \mid a \in C(p')\}$,

$$S(q) = \left\{ Q'^T \begin{pmatrix} a \\ b \end{pmatrix} \mid (a,b) \in S(q') \right\} = \{(Q^T a, b) \mid (a,b) \in S(q')\}$$

and thus $S := \{a \in \mathbb{R}^\ell \mid (a,0) \in S(q)\} = \{Q^T a \mid a \in S'\}$ by Proposition 3.24. Hence $C(p') \subseteq S'$ and $U' \cap C(p') = U' \cap S'$ easily translate into the desired conditions $C(p) \subseteq S$ and $U \cap C(p) = U \cap S$. □

Theorem 8.19. Suppose that Conjecture 8.9 (RZAC) holds. Suppose $k, \ell \in \mathbb{N}_0$. Fix a real zero polynomial $p \in \mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_\ell]$ polynomial of degree $d \geq 2$ and a union W of k many two-dimensional subspaces of $\mathbb{R}^\ell$. For

$$m := k\left(\frac{d(d+1)}{2} - 3\right),$$

there exists a real zero polynomial $q \in \mathbb{R}[x,y] = \mathbb{R}[x_1, \dots, x_\ell, y_1, \dots, y_m]$ of degree d such that $q(x,0) = p$ and for which the spectrahedron

$$S := \{a \in \mathbb{R}^\ell \mid (a,0) \in S(q)\}$$

satisfies

$$C(p) \subseteq S \qquad \text{and} \qquad W \cap C(p) = W \cap S.$$

In particular, $C(p)$ is contained in a spectrahedron that agrees with $C(p)$ on W.

Proof. We prove the theorem for fixed $\ell \in \mathbb{N}_0$ and $p \in \mathbb{R}[x_1, \dots, x_\ell]$ of degree $d \geq 2$ by induction on $k \in \mathbb{N}_0$.

In the base case $k = 0$, we have $W = \emptyset$, $m = 0$, $q = p$, $S = S(q) = S(p)$ so that the claim becomes just $C(p) \subseteq S(p)$ which is Theorem 3.35.

Now for the induction step, let $k \in \mathbb{N}$ and $W = U \cup V$ where U is a two-dimensional subspace of $\mathbb{R}^\ell$ and V is a union of $k-1$ such subspaces.

By induction hypothesis there exists for

$$m' := (k-1)\left(\frac{d(d+1)}{2} - 3\right) = m - \left(\frac{d(d+1)}{2} - 3\right)$$

a real zero polynomial $q' \in \mathbb{R}[x_1, \dots, x_\ell, y_1, \dots, y_{m'}]$ of degree d such that $q'(x,0) = p$ and for which the spectrahedron

$$S' := \{a \in \mathbb{R}^\ell \mid (a,0) \in S(q')\}$$

satisfies

$$C(p) \subseteq S' \qquad \text{and} \qquad V \cap C(p) = V \cap S'.$$

Applying Lemma 8.18 to q' and the two-dimensional subspace $U' := U \times \{0\} \subseteq \mathbb{R}^{\ell+m'}$, we get a real zero polynomial $q \in \mathbb{R}[x,y]$ of degree d such that

$$q(x_1, \dots, x_\ell, y_1, \dots, y_{m'}, 0) = q'$$

and for which the spectrahedron

$$S'' := \{(a,b) \in \mathbb{R}^{\ell+m'} \mid (a,b,0) \in S(q)\} \subseteq \mathbb{R}^{\ell+m'}$$

satisfies

$$C(q') \subseteq S'' \qquad \text{and} \qquad U' \cap C(q') = U' \cap S''.$$

By Lemma 3.26, we have $S'' \subseteq S(q')$ and hence

$$S := \{a \in \mathbb{R}^\ell \mid (a,0) \in S(q)\} = \{a \in \mathbb{R}^\ell \mid (a,0) \in S''\} \subseteq S'.$$

Intersecting the subsets $C(q')$, S'' and U' of $\mathbb{R}^{\ell+m'}$ with $\mathbb{R}^\ell \times \{0\} \subseteq \mathbb{R}^\ell$, we conclude from the already proven that

$$C(p) \subseteq S \qquad \text{and} \qquad U \cap C(q) \subseteq U \cap S.$$

It only remains to show that $W \cap C(p) = W \cap S$. Because of $W = U \cup V$, it suffices to prove that $V \cap C(p) \subseteq V \cap S$. But this follows easily from $V \cap C(p) = V \cap S'$, $S \subseteq S'$ and $C(p) \subseteq S$. □

8.3. **The weak real zero amalgamation conjecture.**

Conjecture 8.20 (weak real zero amalgamation conjecture, WRZAC)**.** Let $\ell, m, n \in \mathbb{N}_0$ and consider $\ell + m + n$ variables coming in three blocks $x = (x_1, \dots, x_\ell)$, $y = (y_1, \dots, y_m)$ and $z = (z_1, \dots, z_n)$. Suppose that $p \in \mathbb{R}[x, y]$ and $q \in \mathbb{R}[x, z]$ are real zero polynomials with

$$p(x, 0) = q(x, 0).$$

Then there exists a real zero polynomial $r \in \mathbb{R}[x, y, z]$ (of no matter what degree) such that the polynomials $r(x, y, 0)$ and p coincide as well as the cubic parts of $r(x, 0, z)$ and q, i.e.,

$$r(x, y, 0) = p \qquad \text{and} \qquad \operatorname{trunc}_3 r(x, 0, z) = \operatorname{trunc}_3 q.$$

Conjecture 8.9 (RZAC) implies of course Conjecture 8.20 (WRZAC). In particular, it is proven for the cases mentioned in Theorem 8.15. Note however that the case $\ell = 0$ is completely trivial: If $p \in \mathbb{R}[y]$ and $q \in \mathbb{R}[z]$ are real zero polynomials with $p(0) = q(0)$, then

$$r := \frac{pq}{p(0)} = \frac{pq}{q(0)} \in \mathbb{R}[y, z]$$

is a real zero polynomial such that $r(y, 0) = p$ and $r(0, z) = q$.

Lemma 8.21. Suppose that Conjecture 8.20 (WRZAC) holds. Let $p \in \mathbb{R}[x_1, x_2, y_1, \dots, y_m] = \mathbb{R}[x, y]$ be a real zero polynomial of degree $d \geq 2$. Set

$$n := \left(\frac{d(d+1)}{2} - 2\right) \in \mathbb{N}_0.$$

Then there exists a real zero polynomial $q \in \mathbb{R}[x, y, z_1, \dots, z_n] = \mathbb{R}[x, y, z]$ such that $q(x, y, 0) = p$,

$$C(p) \subseteq \{(a, b) \in \mathbb{R}^{2+m} \mid (a, b, 0) \in S_\infty(q)\} \qquad \text{and}$$
$$\{a \in \mathbb{R}^2 \mid (a, 0) \in C(p)\} = \{a \in \mathbb{R}^2 \mid (a, 0, 0) \in S_\infty(q)\}.$$

In particular, $C(p)$ is contained in a spectrahedron that agrees with $C(p)$ on the plane spanned by the first two unit vectors.

Proof. WLOG $p(0) = 1$. By the Helton-Vinnikov theorem 2.7, we find symmetric $A_1, A_2 \in \mathbb{R}^{d \times d}$ such that $p(x, 0) = \det(I_d + x_1 A_1 + x_2 A_2)$.

We now claim that we can find real symmetric matrices $B_1, \dots, B_n \in \mathbb{R}^{d \times d}$ such that they span together with A_1 and A_2 a subspace of the space of real symmetric matrices of size d that is perfect in the sense of Definition 3.29. To this end, we distinguish three different cases.

If $A_1 = A_2 = 0$, then we simply set $B_i := 0$ for all $i \in \{1, \ldots, n\}$ since $\{0\} \subseteq \mathbb{R}^{d\times d}$ is trivially perfect.

If the real span of A_1 and A_2 is one-dimensional, then we can jointly diagonalize A_1 and A_2 by conjugating them with a suitable orthogonal matrix and therefore can assume them to be diagonal. Then the co-dimension of the span of A_1 and A_2 inside the space of real diagonal matrices is $d-1 = (d+1)-2$ which is at most n because of $d \geq 2$. Hence we find diagonal matrices $B_1, \ldots, B_n \in \mathbb{R}^{d\times d}$ such that the span of $A_1, A_2, B_1, \ldots, B_n$ is the space of all diagonal matrices which is again perfect.

The last case is where A_1 and A_2 are linearly independent. Then we complete them to a basis $A_1, A_2, B_1, \ldots, B_n$ of the perfect space of real symmetric matrices of size d.

The claim is now proven and we choose $B_1, \ldots, B_n \in \mathbb{R}^{d\times d}$ according to it. By Conjecture 8.20 (WRZAC), we find for

$$r := \det(I_d + x_1A_1 + x_2A_2 + z_1B_1 + \ldots + z_nB_n) \in \mathbb{R}[x, z]$$

a real zero polynomial $q \in \mathbb{R}[x, y, z]$ such that

$$q(x, y, 0) = p \qquad \text{and} \qquad \operatorname{trunc}_3 q(x, 0, z) = \operatorname{trunc}_3 r.$$

By Theorem 3.35 and Remark 3.20, we have $C(q) \subseteq S(q) \subseteq S_\infty(q)$. Together with Lemma 3.26(b), this yields our first statement

$$C(p) = \{(a, b) \in \mathbb{R}^{2+m} \mid (a, b, 0) \in C(q)\} \subseteq \{(a, b) \in \mathbb{R}^{2+m} \mid (a, b, 0) \in S_\infty(q)\}.$$

We have $C(r) = S_\infty(r) = S_\infty(q(x, 0, z))$ by Proposition 3.33(d) and Lemma 3.21. Together with Lemma 3.26, we have the chain of inclusions

$$\begin{aligned}
\{a \in \mathbb{R}^2 \mid (a, 0) \in C(p)\} &= \{a \in \mathbb{R}^2 \mid a \in C(p(x, 0))\} \\
&= \{a \in \mathbb{R}^2 \mid a \in C(r(x, 0))\} = \{a \in \mathbb{R}^2 \mid (a, 0) \in C(r)\} \\
&= \{a \in \mathbb{R}^2 \mid (a, 0) \in S_\infty(q(x, 0, z))\} \supseteq \{a \in \mathbb{R}^2 \mid (a, 0, 0) \in S_\infty(q)\} \\
&\supseteq \{a \in \mathbb{R}^2 \mid (a, 0) \in C(p)\}
\end{aligned}$$

where the last inclusion follows from the already proven part of the lemma. □

Theorem 8.22. Suppose that Conjecture 8.20 (WRZAC) holds. Suppose $k, \ell \in \mathbb{N}_0$. Fix a real zero polynomial $p \in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_\ell]$ polynomial of degree $d \geq 2$ and a union W of k many two-dimensional subspaces of $\mathbb{R}^\ell$. For

$$m := k\left(\frac{d(d+1)}{2} - 2\right),$$

there exists a real zero polynomial $q \in \mathbb{R}[x, y] = \mathbb{R}[x_1, \ldots, x_\ell, y_1, \ldots, y_m]$ (of some degree) such that $q(x, 0) = p$ and for which the spectrahedron

$$S := \{a \in \mathbb{R}^\ell \mid (a, 0) \in S_\infty(q)\}$$

satisfies

$$C(p) \subseteq S \qquad \text{and} \qquad W \cap C(p) = W \cap S.$$

In particular, $C(p)$ is contained in a spectrahedron that agrees with $C(p)$ on W.

Proof. Completely analogous to the proof of Theorem 8.22 where Lemma 8.17 is exchanged by Lemma 8.21. □

8.4. **Tying with a ribbon instead of a cord in the case of cubic real zero polynomials.** In the last two subsections, we showed that we can wrap rigidly convex sets into a spectrahedron and tie them with finitely many cords provided Conjecture 8.9 (RZAC) or at least Conjecture 8.20 (WRZAC) holds true. For rigidly convex sets defined by *cubic* real zero polynomials, we will be able to improve this. Namely, we can even tie by a two-dimensional version of cords, say by ribbons. This means that we can make the spectrahedron agree on finitely many three-dimensional (instead of two-dimensional) subspaces with the rigidly convex set. The technique for the proof is almost literally the same except that we use instead of the Helton-Vinnikov Theorem 2.7 now the following complex version which is a version of the Helton-Vinnikov Corollary 2.8 which allows for one more variable:

Theorem 8.23 (Buckley and Košir)**.** If $p \in \mathbb{R}[x_1, x_2, x_3]$ is a cubic real zero polynomial with $p(0) = 1$, then there exist hermitian matrices $A_1, A_2, A_3 \in \mathbb{C}^{3\times 3}$ such that

$$p = \det(I_3 + x_1 A_1 + x_2 A_2 + x_3 A_3).$$

Proof. Under a certain smoothness assumption, this follows from [BK, Theorem 6.4]. With a perturbation and limit argument, this smoothness assumption can be removed by standard techniques. This is explained in the proof of [Kum1, Proposition 8]. □

The details will be provided in future versions of this article.

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Fachbereich Mathematik und Statistik, Universität Konstanz, 78457 Konstanz, Germany

Email address: `markus.schweighofer@uni-konstanz.de`