PURE STATES, POSITIVE MATRIX POLYNOMIALS AND SUMS OF HERMITIAN SQUARES

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Abstract. Let M be an archimedean quadratic module of real $t \times t$ matrix polynomials in n variables, and let $S \subseteq \mathbb{R}^n$ be the set of all points where each element of M is positive semidefinite. Our key finding is a natural bijection between the set of pure states of M and $S \times \mathbb{P}^{t-1}(\mathbb{R})$. This leads us to conceptual proofs of positivity certificates for matrix polynomials, including the recent seminal result of Hol and Scherer: If a symmetric matrix polynomial is positive definite on S, then it belongs to M. We also discuss what happens for nonsymmetric matrix polynomials or in the absence of the archimedean assumption, and review some of the related classical results. The methods employed are both algebraic and functional analytic.

1. Introduction

We write $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ for the sets of natural, rational, real and complex numbers, respectively. The complex numbers $\mathbb{C}$ always come equipped with the complex-conjugation involution. For any matrix A over a ring with involution $\mathcal{A}$, we denote by A^* its conjugate transpose. If A is a real matrix, A^* is simply its transpose. Let $\operatorname{Sym} \mathcal{A}^{t\times t} := \{A \in \mathcal{A}^{t\times t} \mid A = A^*\}$ be the set of all *symmetric* $t \times t$ matrices. Examples of these include *hermitian squares*, i.e., elements of the form A^*A for some $A \in \mathcal{A}^{t\times t}$.

Recall that a matrix $A \in \mathbb{R}^{t\times t}$ is called *positive semidefinite* if it is symmetric and $\langle Av, v\rangle = v^*Av \geq 0$ for all vectors $v \in \mathbb{R}^t$, A is *positive definite* if it is positive semidefinite and invertible, and is called *negative semidefinite* if $-A$ is positive semidefinite. For matrices A and B of the same size, we write $A \preceq B$ (respectively $A \prec B$) to express that $B - A$ is positive semidefinite (respectively positive definite). Geometrically, $A \in \operatorname{Sym} \mathbb{R}^{t\times t}$ is positive semidefinite if and only if all of its eigenvalues are nonnegative, A is positive definite if and only if all of its eigenvalues are positive, and A is not negative semidefinite if and only if one of its eigenvalues is positive. The following algebraic characterizations are easy to prove:

Proposition 1. *Let $A \in \operatorname{Sym} \mathbb{R}^{t\times t}$.*

(a) *$A \succeq 0$ if and only if A is a sum of hermitian squares in $\mathbb{R}^{t\times t}$;*

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(b) $A \not\preceq 0$ *if and only if there exist* $B_i, C_j \in \mathbb{R}^{t\times t}$ *such that*

$$\sum_i B_i^* A B_i = 1 + \sum_j C_j^* C_j.$$

The main goal of this article is to explain how this proposition extends to *matrix polynomials*, i.e., elements of the ring $\mathbb{R}[\bar{X}]^{t\times t}$ where $\mathbb{R}[\bar{X}]$ is the ring of polynomials in n variables $\bar{X} = (X_1, \dots, X_n)$ with coefficients from $\mathbb{R}$. Note that in $\mathbb{R}^{t\times t}$, every sum of hermitian squares is of course a hermitian square. The reason why we speak of *sums* of hermitian squares in Proposition 1 is that this is no longer true in $\mathbb{R}[\bar{X}]^{t\times t}$. Note however, that $A \in \mathbb{R}[\bar{X}]^{t\times t}$ is a sum of hermitian squares in $\mathbb{R}[\bar{X}]^{t\times t}$ if and only if there is $u \in \mathbb{N}$ and $B \in \mathbb{R}[\bar{X}]^{u\times t}$ such that $A = B^*B$.

Let $\mathcal{A}$ be a ring with *involution* $a \mapsto a^*$ (i.e., $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $a^{**} = a$ for $a, b \in \mathcal{A}$) and set $\operatorname{Sym}\mathcal{A} := \{a \in \mathcal{A} \mid a = a^*\}$. A subset $M \subseteq \operatorname{Sym}\mathcal{A}$ is called a *quadratic module* in $\mathcal{A}$ if

$$1 \in M, \quad M + M \subseteq M \quad \text{and} \quad a^*Ma \subseteq M \text{ for all } a \in \mathcal{A}.$$

To every $G \subseteq \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$, we associate the set

$$S_G := \{x \in \mathbb{R}^n \mid \forall g \in G : \ g(x) \succeq 0\}$$

and the quadratic module M_G generated by G in $\mathbb{R}[\bar{X}]^{t\times t}$. That is,

$$M_G = \left\{ \sum_{i=1}^N p_i^* g_i p_i \mid N \in \mathbb{N}, g_i \in \{1\} \cup G, p_i \in \mathbb{R}[\bar{X}]^{t\times t} \right\}.$$

In particular, $M_\varnothing$ is the set of all sums of hermitian squares in $\mathbb{R}[\bar{X}]^{t\times t}$.

Given a matrix polynomial $f \in \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$ and $S \subseteq \mathbb{R}^n$, we write $f \succeq 0$ on S if for all $x \in S$, $f(x) \succeq 0$. Likewise we use $f \succ 0$, $f \not\preceq 0$. With this notation, $f \in M_G$ implies $f \succeq 0$ on S_G.

In the sequel, we investigate how M_G can be used to describe matrix polynomials $f \in \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$ with $f \succ 0$, $f \succeq 0$ or $f \not\preceq 0$ on S_G. In Section 2, the case $G = \varnothing$ is considered; classical results on globally positive semidefinite matrix polynomials in one or more variables are reviewed, and then we turn to nowhere negative semidefiniteness. We give a sum of hermitian squares representation with denominators in the one variable case and prove mostly negative results for the case of matrix polynomials in several variables.

Our main results are presented in Section 3 which is devoted to the case of *compact* S_G. Actually we work under the slightly stronger assumption that the quadratic module M_G is *archimedean* (which can be enforced by possibly enlarging G without changing S_G). Under this assumption we describe all pure states on $\operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$ (extremal linear forms positive with respect to M_G) as being of the form $p \mapsto \langle p(x)v, v\rangle$ for some $x \in S_G$ and $v \in \mathbb{R}^t$. From this we deduce certificates, in the spirit of Proposition 1, for matrix polynomials being nowhere negative semidefinite on S_G or positive semidefinite on S_G in the spirit of Proposition 1. The latter was originally proved by Hol and Scherer [HS] with entirely different methods.

For a study of positivity of matrix polynomials in *noncommuting* variables, we refer the reader to Helton and McCullough, see e.g. [HM]. Burgdorf, Scheiderer and the second author [BSS] investigate pure states and their role in commutative algebra.

2. Globally positive matrix polynomials

There are various notions of positivity for matrices. Like in Proposition 1, we consider positivity of the smallest and largest eigenvalue, respectively.

2.1. **Globally positive semidefinite matrix polynomials.** By Gauß' theorem, every nonnegative univariate real polynomial is a sum of two squares of real polynomials. The extension to univariate real matrix polynomials was first given by Jakubovič [Ja] and is in a different form commonly known as the Kalman-Jakubovič-Popov lemma [AIP]. It is one of the vast number of matrix factorization results obtained and used in operator and control theory [GKS, GLR, RR]. We refer the reader to [AIP] for a nice algorithmic proof; see also [Dj1].

Theorem 2 (Jakubovič)**.** *For $f \in \operatorname{Sym}\mathbb{R}[Z]^{t\times t}$, the following are equivalent:*

(i) *$f \succeq 0$ on $\mathbb{R}$;*
(ii) *f is a sum of two hermitian squares in $\mathbb{R}[Z]^{t\times t}$.*

Note that (ii) is equivalent to $f = g^*g$ for some $g \in \mathbb{R}[Z]^{2t\times t}$, or $f = \sum_{i=1}^{2t} v_i v_i^*$ for some $v_i \in \mathbb{R}[Z]^t$ (cf. also [CLR, FRS]).

The multivariate version of Gauß' theorem is Artin's solution to Hilbert's 17th problem [Ma, PD]: a nonnegative multivariate real polynomial is a sum of squares of real *rational* functions. A multivariate version of Jakubovič's theorem (and at the same time the matrix version of Artin's theorem) was obtained by Gondard and Ribenboim [GR] in 1974 and reproved several times, e.g. [Dj2, PS, HiN].

Theorem 3 (Gondard & Ribenboim)**.** *For $f \in \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$, the following are equivalent:*

(i) *$f \succeq 0$ on $\mathbb{R}^n$;*
(ii) *$p^2 f$ is a sum of hermitian squares in $\mathbb{R}[\bar{X}]^{t\times t}$ for some nonzero $p \in \mathbb{R}[\bar{X}]$.*

Proof. From (ii) it follows that $f \succeq 0$ on $\{x \in \mathbb{R}^n \mid p(x) \neq 0\}$ and hence (i). Conversely, suppose that (i) holds. By diagonalization of quadratic forms over a field, there exists an invertible matrix $g \in \mathbb{R}(\bar{X})^{t\times t}$ and a diagonal matrix $d \in \mathbb{R}(\bar{X})^{t\times t}$ such that $f = g^*dg$. By (i), d is positive semidefinite where defined. By Artin's solution to Hilbert's 17th problem, we find a nonzero $p \in \mathbb{R}[\bar{X}]$ such that $p^2 d$ is a sum of (hermitian) squares in $\mathbb{R}[\bar{X}]^{t\times t}$. Without loss of generality, we can assume that p^2 also clears the denominators in g. ∎

In the literature cited above, one can find refinements of (ii) at the expense of more complicated proofs, e.g. (ii') $p^2 f$ is a sum of squares in the commutative ring $\mathbb{R}[\bar{X}, f] \subseteq \mathbb{R}[\bar{X}]^{t\times t}$ for some nonzero $p \in \mathbb{R}[\bar{X}]$. Also, Gondard and Ribenboim [GR] prove a bound on the number of hermitian squares needed.

2.2. **Nowhere negative semidefinite matrix polynomials.** We now turn to symmetric nonnegative matrix polynomials which are not negative semidefinite globally, i.e., whose evaluations all have at least one positive eigenvalue. We start by giving an analog of Proposition 1(b) for *univariate* matrix polynomials. Though this is the perfect counterpart to the well known Theorem 2, it is to the best of our knowledge an entirely new result.

Theorem 4. *For* $f \in \operatorname{Sym} \mathbb{R}[Z]^{t\times t}$*, the following are equivalent:*

(i) $f \not\preceq 0$ *on* $\mathbb{R}$*;*
(ii) *there exist* $p_i \in \mathbb{R}[Z]^{t\times t}$ *such that* $\sum_i p_i^* f p_i - 1$ *is a sum of hermitian squares.*

Proof. It is clear that (ii) ⇒ (i), cf. Proposition 1(b).

To prove the converse, suppose first that f is diagonal, say $f = \begin{bmatrix} f_1 & & \\ & \ddots & \\ & & f_t \end{bmatrix}$. By assumption (i), $S_{\{-f_1,\dots,-f_t\}} = \varnothing$ is compact. Since we are in the univariate case, this implies that the quadratic module $M_{\{-f_1,\dots,-f_t\}} \subseteq \mathbb{R}[Z]$ contains all polynomials positive on $S_{\{-f_1,\dots,-f_t\}}$ [PD, Theorem 6.3.8]. In particular, with $f_0 := -1$, there are $g_{ij} \in \mathbb{R}[Z]$ satisfying

$$-1 = \sum_{i=0}^{t} (-f_i) \sum_j g_{ij}^2. \tag{1}$$

Observe that for each i,

$$f_i = \sum_{k=1}^{t} E_{ik}^* f E_{ik} \in M_{\{f\}}, \tag{2}$$

where E_{jk} are the $t \times t$ matrix units. Thus (1) implies

$$\sum_{i,k=1}^{t} \sum_j (E_{ik} g_{ij})^* f (E_{ik} g_{ij}) - 1 = \sum_{i=1}^{t} \sum_j g_{ij}^* f_i g_{ij} - 1 = \sum_j g_{0j}^2 \tag{3}$$

is a sum of hermitian squares.

Now suppose f is not necessarily diagonal. By the version of the LDU decomposition for matrix polynomials given in [Sm2, Proposition 8], there exist diagonal matrices $D_\ell \in \mathbb{R}[Z]^{t\times t}$, and matrices $C_\ell \in \mathbb{R}[Z]^{t\times t}$, $\ell = 1, \dots, m$, satisfying

(a) $D_\ell = C_\ell^* (-f) C_\ell$,
(b) for each $x \in \mathbb{R}$, $-f(x) \succeq 0$ if and only if for all ℓ, $D_\ell(x) \succeq 0$.

From (b) it follows that the diagonal matrix $\begin{bmatrix} -D_1 & & \\ & \ddots & \\ & & -D_m \end{bmatrix}$ is nowhere negative semidefinite. If D_ℓ is the diagonal matrix with entries $d_{j,\ell}$, $j = 1, \dots, t$, then again by [PD, Theorem 6.3.8] we deduce $-1 \in M_{\{d_{j,\ell} \mid j=1,\dots,t,\ \ell=1,\dots,m\}}$. Like in (2) we have $d_{j,\ell} \in M_{\{D_\ell\}} \subseteq M_{\{D_1,\dots,D_m\}}$ for all j, ℓ. Since $M_{\{D_1,\dots,D_m\}} \subseteq M_{\{-f\}}$ by (a), we conclude as in (3) that $-1 \in M_{\{-f\}}$. ∎

Unlike in the univariate case or for Proposition 1(a) where a satisfactory statement on the level of *multivariate* matrix polynomials has been given in the previous subsection (see Theorem 3), there does not seem to exist a straightforward extension of Proposition 1(b) to the multivariate case.

Example 5. Consider $f \in \operatorname{Sym}\mathbb{R}[\bar{X}]^{2\times 2}$. We have $f \not\preceq 0$ on $\mathbb{R}^n$ if and only if $\operatorname{tr} f(x) > 0$ or $\det f(x) < 0$ for all $x \in \mathbb{R}^n$.

(a) Let f be diagonal. Then $f \not\preceq 0$ on $\mathbb{R}^n$ if and only if there exist $p_i \in \mathbb{R}[\bar{X}]^{2\times 2}$ such that $\sum_i p_i^* f p_i \in 1 + M_\varnothing$. Indeed, the implication ($\Leftarrow$) is easy (cf. Proposition 1). For the converse implication, suppose $f = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \not\preceq 0$ on $\mathbb{R}^n$. Then $a > 0$ on $S_{\{-c\}}$ and therefore $p^2 a = 1 + \sigma - \tau c$ for some $p \in \mathbb{R}[\bar{X}]$ and sums of squares $\sigma, \tau \in \mathbb{R}[\bar{X}]$ by Krivine's Positivstellensatz (see, e.g. [Ma, Chapter 2] or [PD, Section 4.2]). Obviously, there exist diagonal $h_i \in \mathbb{R}[\bar{X}]^{2\times 2}$ such that $\sum_i h_i^* f h_i = \begin{bmatrix} p^2 a & 0 \\ 0 & \tau c \end{bmatrix}$. Now

$$\sum_i h_i^* f h_i + \sum_i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^* h_i^* f h_i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+\sigma & 0 \\ 0 & 1+\sigma \end{bmatrix} \in 1 + M_\varnothing.$$

(b) If $f = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, then

$$\begin{bmatrix} 1 & -b \\ 0 & a \end{bmatrix}^* f \begin{bmatrix} 1 & -b \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -c & b \end{bmatrix}^* f \begin{bmatrix} 0 & 1 \\ -c & b \end{bmatrix} = \begin{bmatrix} \operatorname{tr} f & 0 \\ 0 & (\operatorname{tr} f)(\det f) \end{bmatrix}. \tag{4}$$

Unable to settle the general case, we assume $\operatorname{tr} f(x) \neq 0$ for all $x \in \mathbb{R}^n$. Then the diagonal matrix on the right hand side of (4) is nowhere negative semidefinite on $\mathbb{R}^n$. By part (a) above, we obtain $h_i \in \mathbb{R}[\bar{X}]^{2\times 2}$ such that $\sum_i h_i^* f h_i \in 1 + M_\varnothing$.

Example 6. The diagonal matrix

$$f := \begin{bmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_1X_2 + 1 \end{bmatrix} \in \operatorname{Sym}\mathbb{R}[\bar{X}]^{3\times 3}$$

satisfies $f \not\preceq 0$ on $\mathbb{R}^2$ and yet there do not exist $p_i \in \mathbb{R}[\bar{X}]^{2\times 2}$ with $\sum_i p_i^* f p_i \in 1 + M_\varnothing$. This example is inspired by [Ma, Example 7.3.2(i)] which is a modification of the Jacobi-Prestel example [PD, Example 6.3.1].

By way of contradiction, assume $\sum_i p_i^* f p_i = 1 + q$ where $q \in M_\varnothing$. Extracting the top left entry on both sides of this equation, we get sums of squares $\sigma_1, \sigma_2, \sigma_3, \tau \in \mathbb{R}[X_1, X_2]$ with

$$\sigma_1 X_1 + \sigma_2 X_2 + \sigma_3(X_1X_2 + 1) = 1 + \tau.$$

In particular, -1 lies in the quadratic module generated in $\mathbb{R}[X_1, X_2]$ by $-X_1$, $-X_2$ and $-X_1X_2 - 1$, contradicting the existence of a semiordering in $\mathbb{R}[X_1, X_2]$ containing these three polynomials (cf. [Ma, Example 7.3.1]).

We will see that a sum of hermitian squares representation with denominators (and weights) *does* exist for matrix polynomials nonnegative on a compact set with archimedean corresponding quadratic module (see Subsection 3.4). This seems to mimic the situation for polynomials in noncommutative variables studied e.g. in [KS], where a *Nirgendsnegativsemidefinitheitsstellensatz* describing nonnegativity on a bounded set has been given, while the global case is still an open problem; see [KS, Open problem 3.2] for a precise formulation.

3. Archimedean quadratic modules of matrix polynomials

C^*-algebras $\mathcal{A}$ enjoy the following boundedness property: for all $a \in \mathcal{A}$ there is an $N \in \mathbb{N}$ such that $N - a^*a$ is a sum of hermitian squares (actually a hermitian square). In this section, we try to mimic this boundedness property in an algebraic context for other rings with involution. In the rigid context of (matrix) polynomials, sums of hermitian squares have of course to be replaced by a general quadratic module.

3.1. Archimedean quadratic modules.

A quadratic module M of a ring with involution $\mathcal{A}$ is said to be *archimedean* if

$$\forall a \in \mathcal{A}\ \exists N \in \mathbb{N}:\ N - a^*a \in M. \tag{5}$$

To a quadratic module $M \subseteq \operatorname{Sym}\mathcal{A}$ we associate its *ring of bounded elements*

$$H_M(\mathcal{A}) := \{a \in \mathcal{A} \mid \exists N \in \mathbb{N}:\ N - a^*a \in M\}.$$

A quadratic module $M \subseteq \operatorname{Sym}\mathcal{A}$ is thus archimedean if and only if $H_M(\mathcal{A}) = \mathcal{A}$. The name *ring* of bounded elements is justified by the following proposition originally due to Vidav [Vi]; see also [Ci] for a more accessible reference:

Proposition 7 (Vidav)**.** *Let $\mathcal{A}$ be a ring with involution, $\frac{1}{2} \in \mathcal{A}$ and $M \subseteq \operatorname{Sym}\mathcal{A}$ a quadratic module. Then $H_M(\mathcal{A})$ is a subring of $\mathcal{A}$ and is closed under the involution.*

In case $\mathcal{A}$ is an $\mathbb{R}$-algebra, it suffices to check the archimedean condition (5) on a set of algebra generators.

Lemma 8. *A quadratic module $M \subseteq \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$ is archimedean if and only if there exists $N \in \mathbb{N}$ with $N - \sum_i X_i^2 \in M$.*

Proof. The "only if" direction is obvious. For the converse, observe that $\mathbb{R}[\bar{X}]^{t\times t}$ is generated as an $\mathbb{R}$-algebra by $\bar{X}$ and the matrix units E_{ij}, $i, j = 1, \dots, t$. By assumption,

$$N - X_j^2 = (N - \sum_i X_i^2) + \sum_{i\neq j} X_i^2 \in M,$$

so $X_j \in H_M(\mathbb{R}[\bar{X}]^{t\times t})$ for every j. On the other hand, $E_{ij}^*E_{ij} = E_{jj}$ and thus

$$1 - E_{ij}^*E_{ij} = \sum_{k\neq j} E_{kk}^*E_{kk} \in M.$$

Hence by Proposition 7, $H_M(\mathbb{R}[\bar{X}]^{t\times t}) = \mathbb{R}[\bar{X}]^{t\times t}$ so M is archimedean. ∎

3.2. **Pure states.** In functional analysis, the concept of pure states is well-established, see e.g. [Ar, Sections 1.6 and 1.7] for the classical application to C^*-algebras and their representations. Here we adopt these ideas to matrix polynomials.

Let $G \subseteq \operatorname{Sym} \mathbb{R}[\bar{X}]^{t\times t}$. A linear form $L : \operatorname{Sym} \mathbb{R}[\bar{X}]^{t\times t} \to \mathbb{R}$ is called a *state* on $(\operatorname{Sym} \mathbb{R}[\bar{X}]^{t\times t}, M_G)$ if $L(M_G) \subseteq \mathbb{R}_{\geq 0}$ and $L(1) = 1$. A state L is called *pure* if it is an extreme point of the convex set of all states, i.e., it is not a proper convex combination of two states other than L.

We now come to the central result of this article. It is a matrix polynomial version of the well-known theorem stating that for every pure state L on a C^*-algebra $\mathcal{A}$ there exists a unit vector v in a Hilbert space $\mathcal{H}$ and an irreducible $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $L(a) = \langle \pi(a)v, v\rangle$ for all $a \in \mathcal{A}$ (see e.g. [Ar, Theorem 1.6.6]).

Theorem 9. *Suppose $G \subseteq \operatorname{Sym} \mathbb{R}[\bar{X}]^{t\times t}$ and M_G is archimedean. For each pure state L on $(\operatorname{Sym} \mathbb{R}[\bar{X}]^{t\times t}, M_G)$, there exists $x \in S_G$ and a unit vector $v \in \mathbb{R}^t$ such that*

$$L(p) = \langle p(x)v, v\rangle \qquad \text{for all } p \in \operatorname{Sym} \mathbb{R}[\bar{X}]^{t\times t}.$$

Proof. We extend L to $\mathbb{C}[\bar{X}]^{t\times t}$ by setting

$$L(p + \mathbb{i}q) = \frac{1}{2}(L(p + p^*) + \mathbb{i}L(q + q^*)) \tag{6}$$

for $p, q \in \mathbb{R}[\bar{X}]^{t\times t}$. This is the unique $\mathbb{C}$-linear extension of L satisfying $L(f^*) = L(f)^*$ for all $f \in \mathbb{C}[\bar{X}]^{t\times t}$. Let

$$M_G^{\mathbb{C}} := \left\{ \sum_{j=1}^{N} p_j^* g_j p_j \mid N \in \mathbb{N}, g_j \in \{1\} \cup G, p_j \in \mathbb{C}[\bar{X}]^{t\times t} \right\}$$

be the quadratic module generated by G in $\mathbb{C}[\bar{X}]^{t\times t}$. Then L is nonnegative on $M_G^{\mathbb{C}}$. Indeed, given $f = (p + \mathbb{i}q)^* g(p + \mathbb{i}q)$ with $p, q \in \mathbb{R}[\bar{X}]^{t\times t}$ and $g \in \{1\} \cup G$, we have $f = (p^*gp + q^*gq) + \mathbb{i}(p^*gq - q^*gp)$. Applying the definition (6) of L, we obtain $L(f) = L(p^*gp + q^*gq) \in L(M_G) \subseteq \mathbb{R}_{\geq 0}$, as desired. For later use let us observe that $M_G^{\mathbb{C}}$ is archimedean: write $f \in \mathbb{C}[\bar{X}]^{t\times t}$ as $f = f_1 + \mathbb{i}f_2$ with $f_j \in \mathbb{R}[\bar{X}]^{t\times t}$. Then $f_j \in H(M_G) \subseteq H(M_G^{\mathbb{C}})$ and $\mathbb{i} \in H(M_G^{\mathbb{C}})$. Hence Proposition 7 implies $H(M_G^{\mathbb{C}}) = \mathbb{C}[\bar{X}]^{t\times t}$.

By the Cauchy-Schwarz inequality for semi-scalar products,

$$J := \{f \in \mathbb{C}[\bar{X}]^{t\times t} \mid L(f^*f) = 0\} \tag{7}$$

is a linear subspace of $\mathbb{C}[\bar{X}]^{t\times t}$. Similarly, we see that

$$\langle \overline{p}, \overline{q}\rangle := L(q^*p) \tag{8}$$

defines a scalar product on $\mathbb{C}[\bar{X}]^{t\times t}/J$, where $\overline{p} := p + J$ denotes the residue class of $p \in \mathbb{C}[\bar{X}]^{t\times t}$ modulo J. Let $\mathcal{H}$ denote the completion of $\mathbb{C}[\bar{X}]^{t\times t}/J$ with respect to this scalar product. Note that $\mathcal{H} \neq \{0\}$ since $1 \notin J$.

We proceed to show J is a left ideal of $\mathbb{C}[\bar{X}]^{t\times t}$. Let $f \in \mathbb{C}[\bar{X}]^{t\times t}$. Since $M_G^{\mathbb{C}}$ is archimedean, there is some $N \in \mathbb{N}$ with $N - f^*f \in M_G^{\mathbb{C}}$. Hence for all $p \in \mathbb{C}[\bar{X}]^{t\times t}$, we have

$$0 \leq L(p^*(N - f^*f)p) \leq NL(p^*p). \tag{9}$$

This shows that $L(p^* f^* f p) = 0$ for all $p \in J$, i.e., $fp \in J$.

Because J is a left ideal, the map

$$\pi : \mathbb{C}[\bar{X}]^{t\times t} \to \mathcal{B}(\mathcal{H}),\ f \mapsto (\overline{p} \mapsto \overline{fp}) \tag{10}$$

is well-defined. Here $\overline{p} \mapsto \overline{fp}$ stands for the unique bounded linear extension to $\mathcal{H}$ of the left multiplication with $\overline{f}$ on $\mathbb{C}[\bar{X}]^{t\times t}/J$, which is well-defined by (9). Using the definition (8) of the scalar product, it is easy to see that π is a homomorphism of rings with involution, i.e., a $*$-representation of $\mathbb{C}[\bar{X}]^{t\times t}$ on $\mathcal{H}$. Setting $w := \overline{1} \in \mathcal{H}$, we observe that

$$L(f) = \langle \pi(f)w, w\rangle \tag{11}$$

for all $f \in \mathbb{C}[\bar{X}]^{t\times t}$.

We claim that the commutant $\pi(\mathbb{C}[\bar{X}]^{t\times t})'$ of the image of π in $\mathcal{B}(\mathcal{H})$ is $\mathbb{C}$. To see this, we take an arbitrary operator $T \in \pi(\mathbb{C}[\bar{X}]^{t\times t})'$. Since the commutant is closed under the involution and $T = \frac{T+T^*}{2} + \mathbb{i}\frac{T-T^*}{2\mathbb{i}}$, we are reduced to the case $T = T^*$. By the spectral theorem, T decomposes into projections belonging to $\{T\}'' \subseteq \pi(\mathbb{C}[\bar{X}]^{t\times t})'$. So we can even assume T is a projection. By way of contradiction, assume $T \neq 0$ and $T \neq 1$. Since $T \in \pi(\mathbb{C}[\bar{X}]^{t\times t})'$ and w is a cyclic vector for π by construction, it follows that $Tw \neq 0$ and $(1-T)w \neq 0$. This allows us to define states L_i on $(\operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}, M_G)$ by

$$L_1(f) = \frac{\langle \pi(f)Tw, Tw\rangle}{\|Tw\|^2} \qquad \text{and} \qquad L_2(f) = \frac{\langle \pi(f)(1-T)w, (1-T)w\rangle}{\|(1-T)w\|^2}$$

for all $f \in \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$. One checks that L is a convex combination of L_1 and L_2. The state L being pure, we obtain $L = L_i$. By (11), this implies

$$\langle \pi(f)w, \lambda w\rangle = \lambda\langle \pi(f)w, w\rangle = \langle \pi(f)Tw, Tw\rangle = \langle T\pi(f)w, Tw\rangle = \langle \pi(f)w, Tw\rangle$$

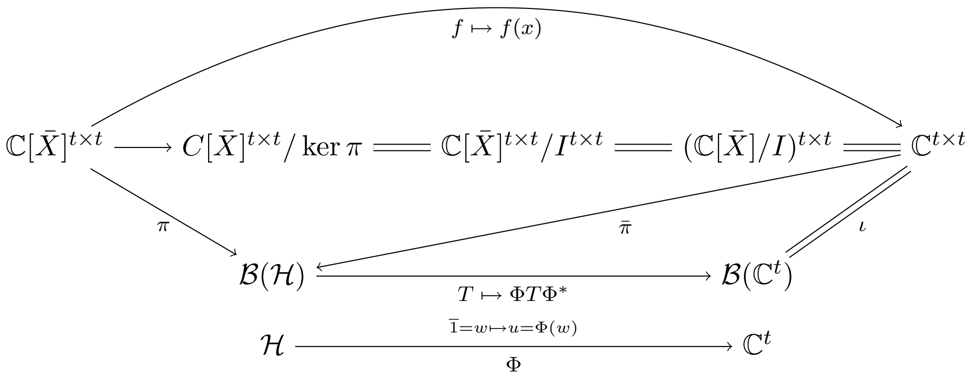
for all $f \in \mathbb{C}[\bar{X}]^{t\times t}$, where $\lambda := \|Tw\|^2$. In particular, $Tw = \lambda w$ since w is a cyclic vector for π. This implies $\lambda \in \{0, 1\}$ since T is a projection, a contradiction.

By [La, Theorem 3.1], $\ker\pi = I^{t\times t}$ for an ideal I of $\mathbb{C}[\bar{X}]$. Since $\ker\pi$ is closed under the involution, I is closed under complex conjugation. Moreover $\mathbb{C}[\bar{X}]/I$ is contained in the center of $(\mathbb{C}[\bar{X}]/I)^{t\times t} = \mathbb{C}[\bar{X}]^{t\times t}/\ker\pi \cong \pi(\mathbb{C}[\bar{X}]^{t\times t})$ which is $\mathbb{C}$ by the above. Hence $\mathbb{C}[\bar{X}]/I = \mathbb{C}$ and $\pi(\mathbb{C}[\bar{X}]^{t\times t}) \cong \mathbb{C}^{t\times t}$ as a C^*-algebra. In particular, there exists $x \in \mathbb{C}^n$ such that $I = \{p \in \mathbb{C}[\bar{X}] \mid p(x) = 0\}$. Actually $x \in \mathbb{R}^n$ since $I = I^*$. Also, $\mathcal{H} = \pi(\mathbb{C}[\bar{X}]^{t\times t})w$ is finite-dimensional.

Next we claim that π is an irreducible $*$-representation. Indeed, suppose U is a linear subspace of $\mathcal{H}$ invariant under every $\pi(f)$ for $f \in \mathbb{C}[\bar{X}]^{t\times t}$. Let $P : \mathcal{H} \to U$ denote the orthogonal projection. It suffices to show that $P \in \pi(\mathbb{C}[\bar{X}]^{t\times t})' = \mathbb{C}$, i.e., $\pi(f)P = P\pi(f)$ for each $f \in \mathbb{C}[\bar{X}]^{t\times t}$. By the standard trick, we reduce to the case $f = f^*$. But then

$$\pi(f)P = P\pi(f)P = (P\pi(f)P)^* = (\pi(f)P)^* = P\pi(f).$$

As indicated in the commutative diagram below, π now induces an irreducible $*$-representation $\bar{\pi}$ of $\mathbb{C}^{t\times t}$. This representation is unitarily equivalent to the identity representation ι [Ar, Corollary 2 to Theorem 1.4.4], i.e., there is a unitary map $\Phi : \mathcal{H} \to \mathbb{C}^t$ making the diagram below commute.



Let $u := \Phi(w) \in \mathbb{C}^t$. For each $p \in \mathbb{C}[\bar{X}]^{t\times t}$, we have

$$\begin{aligned} L(p) &= \langle \pi(p)w, w\rangle = \langle \Phi\pi(p)w, \Phi w\rangle = \langle \Phi\pi(p)\Phi^* u, u\rangle \\ &= \langle \Phi\bar{\pi}(p(x))\Phi^* u, u\rangle = \langle \iota(p(x))u, u\rangle = \langle p(x)u, u\rangle. \end{aligned} \tag{12}$$

In particular, we get $\operatorname{tr}(Auu^*) = \langle Au, u\rangle = L(A) \in \mathbb{R}$ for all $A \in \mathbb{R}^{t\times t}$. This implies $uu^* \in \mathbb{R}^{t\times t}$, that is, uu^* is a real positive semidefinite rank one matrix and can thus be factorized as $uu^* = vv^*$ for some $v \in \mathbb{R}^t$. We can now rewrite (12) as

$$L(p) = \langle p(x)u, u\rangle = \operatorname{tr}(p(x)uu^*) = \operatorname{tr}(p(x)vv^*) = \langle p(x)v, v\rangle$$

for all $p \in \mathbb{R}[\bar{X}]^{t\times t}$. Also note that $\langle v, v\rangle = L(1) = 1$, i.e., v is a unit vector.

It remains to show that $x \in S_G$. To show this, let $g \in G$ and $z \in \mathbb{R}^t$. Choose $A \in \mathbb{R}^{t\times t}$ with $z = Av$. Then

$$\langle g(x)z, z\rangle = \langle g(x)Av, Av\rangle = \langle A^* g(x)Av, v\rangle = L(A^* gA) \geq 0$$

since $A^* gA \in M_G$. ∎

Proposition 10. *Let $G \subseteq \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$. For each $x \in S_G$ and each unit vector $v \in \mathbb{R}^t$, the state L on $(\operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}, M_G)$ defined by $L(p) = \langle p(x)v, v\rangle$ is pure.*

Proof. For convenience of notation, set $L_0 := L$. Suppose there are states L_1 and L_2 on $(\operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}, M_G)$ such that $2L_0 = L_1 + L_2$. Extending each L_i to $\mathbb{C}[\bar{X}]^{t\times t}$ as in (6), this equality still holds. Define linear subspaces $J_i \subseteq \mathbb{C}[\bar{X}]^{t\times t}$ as in (7) by $J_i := \{f \in \mathbb{C}[\bar{X}]^{t\times t} \mid L_i(f^* f) = 0\}$. Obviously

$$J_1 \cap J_2 = J_0 = \{f \in \mathbb{C}[\bar{X}]^{t\times t} \mid f(x)v = 0\}. \tag{13}$$

In particular, $\mathbb{C}[\bar{X}]^{t\times t}/J_0 \cong \mathbb{C}^t$ as vector spaces. Hence $\mathcal{H}_i := \mathbb{C}[\bar{X}]^{t\times t}/J_i$ is of dimension at most t. The GNS construction for L_i yields a scalar product $\langle ., . \rangle_i$ on $\mathcal{H}_i$ defined as in (8) and a $*$-representation $\pi_i : \mathbb{C}[\bar{X}]^{t\times t} \to \mathcal{B}(\mathcal{H}_i)$ (cf. the proof of the previous theorem). Again by [La, Theorem 3.1], there are ideals $I_i \subsetneq \mathbb{C}[\bar{X}]$ such that $\ker\pi_i = I_i^{t\times t}$ and therefore $\mathbb{C}[\bar{X}]^{t\times t}/\ker\pi_i \cong (\mathbb{C}[\bar{X}]/I_i)^{t\times t}$. In particular,

$$\dim \mathcal{B}(\mathcal{H}_i) \geq \dim(\mathbb{C}[\bar{X}]^{t\times t}/\ker\pi_i) \geq t^2 \dim(\mathbb{C}[\bar{X}]/I_i) \geq t^2$$

whence $\dim \mathcal{H}_i \geq t$ and therefore $\dim \mathcal{H}_i = t$. Now by (13), we have $J_0 = J_1 = J_2$. Therefore we have three scalar products on $\mathcal{H} := \mathcal{H}_0 = \mathcal{H}_1 = \mathcal{H}_2$, and we find positive definite matrices $G_1, G_2 \in \mathbb{C}^{t \times t}$ such that

$$L_i(q^* p) = \langle \overline{p}, \overline{q} \rangle_i = v^* q(x)^* G_i p(x) v$$

where $\overline{p} = p(x)v$ denotes the residue class of p modulo J_i (this is also true for $i = 0$ with G_0 being the identity matrix). Since

$$v^* C^* G_i ABv = \langle \overline{AB}, \overline{C} \rangle_i = L_i(C^* AB) = \langle \overline{B}, \overline{A^* C} \rangle_i = v^* C^* A G_i Bv$$

for all $A, B, C \in \mathbb{C}^{t \times t}$, it follows that $G_i A = A G_i$ for all $A \in \mathbb{C}^{t \times t}$, i.e., $G_i \in \mathbb{C}$. More precisely, $G_i = G_i \langle v, v \rangle = v^* G_i v = L_i(1) = 1$. Thus $L = L_1 = L_2$. ∎

For $0 \neq v \in \mathbb{R}^t$, denote by $[v]$ the linear subspace spanned by v seen as an element of the real projective space $\mathbb{P}^{t-1}(\mathbb{R})$ of dimension $t - 1$.

Corollary 11. *Suppose $G \subseteq \operatorname{Sym} \mathbb{R}[\bar{X}]^{t \times t}$ and M_G is archimedean. We have a bijection between $S_G \times \mathbb{P}^{t-1}(\mathbb{R})$ and the set of pure states on* $(\operatorname{Sym} \mathbb{R}[\bar{X}]^{t \times t}, M_G)$ *well-defined by associating to each $(x, [v]) \in S_G \times \mathbb{P}^{t-1}(\mathbb{R})$, v a unit vector of $\mathbb{R}^t$, the map $p \mapsto \langle p(x)v, v \rangle$.*

Proof. By Proposition 10 and $\langle p(x)v, v \rangle = \langle p(x)(-v), -v \rangle$, the map is well-defined. It is surjective by Theorem 9. To show that it is injective, let $(x, [v]), (y, [w]) \in S_G \times \mathbb{P}^{t-1}(\mathbb{R})$ with unit vectors $v, w \in \mathbb{R}^t$ satisfy

$$\langle p(x)v, v \rangle = \langle p(y)w, w \rangle \tag{14}$$

for all $p \in \operatorname{Sym} \mathbb{R}[\bar{X}]^{t \times t}$. Using (14) with $p = E_{ij} + E_{ji}$ yields $vv^* = ww^*$. Then $[v] = [w]$ since $(v^* w)v = vv^* w = ww^* w = w$. Setting $p = X_i$ in (14), we get moreover $x_i = y_i$ whence $x = y$. ∎

In general, Theorem 9 and Corollary 11 fail badly for nonarchimedean M_G.

Example 12. Take $G = \varnothing$, $t = 1$ and $n = 2$, i.e., consider pure states on $(\mathbb{R}[X, Y], M_\varnothing)$ where X and Y are two variables and $M_\varnothing$ is the cone of sums of squares of polynomials. We endow the (algebraic) dual $\mathbb{R}[X, Y]^\vee$ with the weak* topology and consider the closed convex cone $M_\varnothing^\vee \subseteq \mathbb{R}[X, Y]^\vee$ of linear forms $L : \mathbb{R}[X, Y] \to \mathbb{R}$ with $L(M_\varnothing) \subseteq \mathbb{R}_{\geq 0}$. Note that each $0 \neq L \in M_\varnothing^\vee$ becomes a state on $(\mathbb{R}[X, Y], M_\varnothing)$ after multiplication with a positive scalar (for $L(1) = 0$ implies $L = 0$ by a Cauchy-Schwarz argument). Choose a polynomial f with $f \geq 0$ on $\mathbb{R}^2$ and $f \notin M_\varnothing$, for instance the Motzkin polynomial $f = X^2 Y^4 + X^4 Y^2 - 3X^2 Y^2 + 1$ [Ma, Proposition 1.2.2]. Since $M_\varnothing$ is closed in $\mathbb{R}[X, Y]$ with respect to the finest locally convex topology (see, e.g., [Ma, Proposition 4.1.2(2)] together with [Ma, Example 4.1.5]), the Hahn-Banach separation theorem [Ba, Theorem III.3.4] yields $L_0 \in M_\varnothing^\vee$ with $L_0(f) < 0$. As explained above, we can assume that L_0 is a state on $(\mathbb{R}[X, Y], M_\varnothing)$.

Fix a double sequence $(c_{ij})_{i,j \in \mathbb{N}_0}$ of $c_{ij} > 0$ satisfying $\sum_{i,j} c_{ij} L_0(X^{2i} Y^{2j}) = 1$. Now set

$$C := \left\{ L \in M_\varnothing^\vee \mid \sum_{i,j \in \mathbb{N}_0} c_{ij} L(X^{2i} Y^{2j}) \leq 1 \right\}.$$

Then C is weak* closed since $C = \bigcap_{k\in\mathbb{N}_0}\{L \in M_\varnothing^\vee \mid \sum_{i,j=0}^k c_{ij}L(X^{2i}Y^{2j}) \leq 1\}$ and compact (for if $L \in C$, then $|L(X^{2i}Y^{2j})| \leq \frac{1}{c_{ij}}$ for all $i, j \in \mathbb{N}_0$, and this implies by a Cauchy-Schwarz argument similar a priori bounds for the values of L on the other polynomials). In addition, both C and $M_\varnothing^\vee \setminus C$ are obviously convex. Hence C is a *cap* of $M_\varnothing^\vee$ containing L_0 (see [Ph, page 80]).

By the Krein-Milman theorem [Ba, Theorem III.4.1], there exists an extreme point L of C such that $L(f) < 0$. By Choquet theory [Ph, Proposition 13.1], L lies on an extreme ray of $M_\varnothing^\vee$. After normalization, L is a pure state on $(\mathbb{R}[X, Y], M_\varnothing)$. Since $L(f) < 0$ and $f \geq 0$ on $\mathbb{R}^2$, L cannot be a point evaluation.

3.3. **Positive semidefinite matrix polynomials.** Now we are ready to give a version of Proposition 1(a) for matrix polynomials in the archimedean case, originally due to Hol and Scherer [HS, Corollary 1]. Using the above classification of pure states, the proof reduces to an easy separation argument. In contrast to this, the original proof of Hol and Scherer is more involved.

Theorem 13 (Hol & Scherer). *Suppose* $G \cup \{f\} \subseteq \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$ *and* M_G *is archimedean. If* $f \succ 0$ *on* S_G*, then* $f \in M_G$.

Proof. In the terminology of Barvinok [Ba, Definition III.1.6], Proposition 7 together with the identity $4s = (s+1)^2 - (s-1)^2$ shows that M_G is an archimedean quadratic module if and only if 1 is an algebraic interior point of the convex cone $M_G \subseteq \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$. Recall: f is an *algebraic interior point* of M_G if for every $p \in \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$ there exists $\varepsilon > 0$ with $f + \varepsilon p \in M_G$.

Suppose $f \notin M_G$. We will find $x \in S_G$ such that $f(x) \not\succ 0$. The existence of an algebraic interior point of M_G allows us to separate the convex sets M_G and $\mathbb{R}_{>0}f$ by the Eidelheit-Kakutani separation theorem [Ba, Theorem III.1.7]. More precisely, there exists a state L on $(\operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}, M_G)$ with $L(f) \leq 0$. The set of all such states is weak* compact by Tikhonov's theorem (cf. the proof of Alaoglu's theorem [Ba, Theorem III.2.9]). Hence by the Krein-Milman theorem [Ba, Theorem III.4.1], L can be chosen to be pure.

By Theorem 9, there exists $x \in S_G$ and a unit vector $v \in \mathbb{R}^t$ such that $L(p) = \langle p(x)v, v\rangle$ for all $p \in \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$. In particular, $\langle f(x)v, v\rangle = L(f) \leq 0$ as desired. ■

Corollary 14. *Suppose* $G \subseteq \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$ *and* M_G *is archimedean. For* $f \in \operatorname{Sym}\mathbb{R}[\bar{X}]^{t\times t}$*, the following are equivalent:*

(i) $f \succeq 0$ *on* S_G*;*
(ii) $f + \varepsilon \in M_G$ *for all* $\varepsilon \in \mathbb{R}_{>0}$.

For $t = 1$, Theorem 13 specializes to Putinar's Positivstellensatz [Pu]. To avoid possible confusion, we use the letter Q to denote quadratic modules in commutative rings with trivial involution. For instance, if $G \subseteq \mathbb{R}[\bar{X}]$ we denote the quadratic module generated by G in $\mathbb{R}[\bar{X}]$ by Q_G, i.e.,

$$Q_G = \left\{ \sum_{i=1}^N p_i^2 g_i \mid N \in \mathbb{N}, g_i \in \{1\} \cup G, p_i \in \mathbb{R}[\bar{X}] \right\}.$$

Note that $Q_G = M_G$ for $t = 1$ but $Q_G \subsetneq M_G$ for $t > 1$.

Corollary 15 (Putinar)**.** *Suppose $G \cup \{f\} \subseteq \mathbb{R}[\bar{X}]$ and Q_G is archimedean. If $f > 0$ on S_G, then $f \in Q_G$.*

Clearly, if Q_G is archimedean then S_G is compact. The converse is false even for finite $G \subseteq \mathbb{R}[\bar{X}]$ as shown by the Jacobi-Prestel example [PD, Example 6.3.1]. Nevertheless, there is an intimate connection between compactness and the archimedean property established by Schmüdgen [Sm1]. To describe his result, we introduce the following notation: Given a set $G = \{g_1, \dots, g_m\} \subseteq \mathbb{R}[\bar{X}]$ of m distinct polynomials, let $\widehat{G} = \{g_1^{\delta_1} \cdots g_m^{\delta_m} \mid 0 \neq \delta \in \{0,1\}^m\}$ denote the set of the $2^m - 1$ nontrivial products of the g_i.

Theorem 16 (Schmüdgen)**.** *Suppose $G \subseteq \mathbb{R}[\bar{X}]$ is finite. Then S_G is compact if and only if the* (*multiplicative*) *quadratic module $Q_{\widehat{G}}$ is archimedean.*

As an important special case, we obtain that for a singleton $G = \{g\} \subseteq \mathbb{R}[\bar{X}]$, S_G is compact if and only if Q_G is archimedean. This continues to hold if G has exactly two elements [PD, Corollary 6.3.7]. For this and other nontrivial strengthenings of Schmüdgen's theorem due to Jacobi and Prestel we refer to [PD, Chapter 6]. These results allow us to deduce that M_G is archimedean in such cases by the following proposition.

Proposition 17. *If $G \subseteq \mathbb{R}[\bar{X}]$, then Q_G is archimedean if and only if M_G is archimedean.*

Proof. To prove the nontrivial direction, suppose that M_G is archimedean, i.e., $N - \sum_{i=1}^n X_i^2 = \sum_j p_j^* p_j g_j$ for some $N \in \mathbb{N}$, $p_j \in \mathbb{R}[\bar{X}]^{t\times t}$ and $g_j \in G \cup \{1\}$. Since the trace of a hermitian square $p_j^* p_j$ is a sum of squares in $\mathbb{R}[\bar{X}]$, it follows that $N - \sum_{i=1}^n X_i^2 = \frac{1}{t} \sum_j \operatorname{tr}(p_j^* p_j) g_j \in Q_G$. Hence Q_G is archimedean by Lemma 8. ∎

There does not seem to exist a viable generalization of Schmüdgen's theorem for general finite $G \subseteq \operatorname{Sym} \mathbb{R}[\bar{X}]^{t\times t}$. It does not make sense to consider $\widehat{G}$ because products of positive semidefinite matrices are not symmetric in general, let alone positive semidefinite. If G is a singleton, we have $\widehat{G} = G$ but still S_G compact does not imply M_G archimedean.

Example 18. Let $f \in \operatorname{Sym} \mathbb{R}[\bar{X}]^{3\times 3}$ be the diagonal matrix from Example 6. Then $S_{\{-f\}} = \varnothing$ is compact but $M_{\{-f\}}$ is not archimedean. Otherwise Theorem 13 would imply $-1 \in M_{\{-f\}}$ which is not the case as seen in Example 6.

We now briefly turn to positivity of *not necessarily symmetric* matrix polynomials. For this we need the following classical lemma [Br, Section 6.3] (see also [Sw, Theorem 5.3]).

Lemma 19 (Brumfiel)**.** *Let $\mathcal{R}$ be a commutative $\mathbb{Q}$-algebra and $Q \subseteq \mathcal{R}$ a quadratic module. Then $H_Q(\mathcal{R})$ is integrally closed in $\mathcal{R}$.*

Let $G \subseteq \mathbb{R}[\bar{X}]$ and $f \in \mathbb{R}[\bar{X}]^{t\times t}$. The quadratic module generated by G in the commutative ring $\mathbb{R}[\bar{X}, f]$ endowed with the *trivial* involution will be denoted by Q_G^f. Observe that $Q_G^f \subseteq M_G$ if and only if $f = f^*$.

Theorem 20. *Suppose $G \subseteq \mathbb{R}[\bar{X}]$, Q_G is archimedean and $f \in \mathbb{R}[\bar{X}]^{t\times t}$. If for all $x \in S_G$, all* real *eigenvalues of $f(x)$ are positive, then $f \in Q_G^f$.*

Proof. Let $q_f \in \mathbb{R}(\bar{X})[Y]$ be the minimal polynomial of the matrix f. Note that $q_f \in \mathbb{R}[\bar{X}, Y]$ by Gauß' lemma since q_f divides the (monic) characteristic polynomial of f by the Cayley-Hamilton theorem.

Now

$$Y > 0 \quad \text{on} \quad \{(x,y) \in S_G \times \mathbb{R} \mid q_f(x,y) = 0\} = S_{G\cup\{q_f,-q_f\}}.$$

We claim that $Q_{G\cup\{q_f,-q_f\}} = Q_G^Y + \mathbb{R}[\bar{X}, Y]q_f$ is archimedean, or equivalently, the quadratic module Q_G^f is archimedean in $\mathbb{R}[\bar{X}, f] = \mathbb{R}[\bar{X}, Y]/(q_f)$. Indeed, since f is integral over $\mathbb{R}[\bar{X}]$ and $H_{Q_G^f}(\mathbb{R}[\bar{X}, f]) \supseteq H_{Q_G}(\mathbb{R}[\bar{X}]) = \mathbb{R}[\bar{X}]$ is integrally closed, we have $H_{Q_G^f}(\mathbb{R}[\bar{X}, f]) = \mathbb{R}[\bar{X}, f]$.

By Corollary 15, $Y \in Q_{G\cup\{q_f,-q_f\}}$. Plugging in f for Y yields $f \in Q_G^f$. ■

Corollary 21. *Suppose $G \subseteq \mathbb{R}[\bar{X}]$ and Q_G is archimedean. For $f \in \mathbb{R}[\bar{X}]^{t\times t}$, the following are equivalent:*

(i) *for all $x \in S_G$, all* real *eigenvalues of $f(x)$ are nonnegative;*
(ii) *$f + \varepsilon \in Q_G^f$ for all $\varepsilon \in \mathbb{R}_{>0}$.*

Proof. (i)⇒(ii) follows from Theorem 20. For the converse, it suffices to observe that for $p \in Q_G^p$, all real eigenvalues of $p(x)$ are nonnegative for all $x \in S_G$. Indeed, suppose $p = \sum_j h_j(\bar{X}, p)^2 g_j$ for finitely many $g_j \in G \cup \{1\}$ and $h_j \in \mathbb{R}[\bar{X}, Y]$. Let $\lambda \in \mathbb{R}$, $0 \neq v \in \mathbb{R}^t$ and $p(x)v = \lambda v$. Then

$$\lambda v = p(x)v = \sum_j h_j(x, p(x))^2 g_j(x) v = \sum_j h_j(x, \lambda)^2 g_j(x) v$$

whence $\lambda = \sum_j h_j(x,\lambda)^2 g_j(x) \geq 0$. ■

3.4. **Matrix polynomials not negative semidefinite.** We conclude this article with an application of Theorem 13 yielding a version of Proposition 1(b) for matrix polynomials.

Corollary 22 (Matrizenpolynomnirgendsnegativsemidefinitheitsstellensatz)**.** *Suppose $G \subseteq \operatorname{Sym} \mathbb{R}[\bar{X}]^{t\times t}$ and M_G is archimedean. For $f \in \operatorname{Sym} \mathbb{R}[\bar{X}]^{t\times t}$, the following are equivalent:*

(i) *$f \not\preceq 0$ on S_G;*
(ii) *there exist $p_i \in \mathbb{R}[\bar{X}]^{t\times t}$ such that*

$$\sum_i p_i^* f p_i \in 1 + M_G.$$

Proof. (ii)⇒(i) is immediate from Proposition 1(b). For the converse, note that $f \not\preceq 0$ on S_G if and only if $S_{G\cup\{-f\}} = \varnothing$. In this case, $-1 \succ 0$ on $S_{G\cup\{-f\}}$. Since M_G and therefore $M_{G\cup\{-f\}}$ is archimedean, Theorem 13 implies that $-1 \in M_{G\cup\{-f\}}$ which is exactly what we need. ■

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