CERTIFICATES FOR NONNEGATIVITY OF POLYNOMIALS WITH ZEROS ON COMPACT SEMIALGEBRAIC SETS

MARKUS SCHWEIGHOFER

Abstract. We prove a criterion for an element of a commutative ring A to be contained in an archimedean semiring $T \subset A$. It can be used to investigate the question whether nonnegativity of a polynomial on a compact semialgebraic set can be certified in a certain way. In case of (strict) positivity instead of nonnegativity, our criterion simplifies to classical results of Stone, Kadison, Krivine, Handelman, Schmüdgen et al. As an application of our result, we give a new proof of the following result of Handelman: If an odd power of a real polynomial in several variables has only nonnegative coefficients, then so do all sufficiently high powers.

1. Introduction

We write $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ for the sets of natural, integer, rational and real numbers. We use the usual notation for intervals, e.g., $[a, b)$, $[0, \infty)$, $(0, \infty)$ for the interval $\{x \in \mathbb{R} \mid a \leq x < b\}$, the nonnegative and positive real numbers, respectively.

Let $\bar{X} := (X_1, \dots, X_n)$ be a tuple of $n \geq 1$ variables and $\mathbb{R}[\bar{X}] := \mathbb{R}[X_1, \dots, X_n]$ the ring of real polynomials in these variables. Given $p_1, \dots, p_s \in \mathbb{R}[\bar{X}]$, we write

$$\bar{p}^\alpha := p_1^{\alpha_1} \cdots p_s^{\alpha_s}$$

for $\alpha \in \mathbb{N}^s$, and call sets of the form

$$\{p_1 \geq 0, \dots, p_s \geq 0\} := \{x \in \mathbb{R}^n \mid p_1(x) \geq 0, \dots, p_s(x) \geq 0\} \subset \mathbb{R}^n$$

basic closed semialgebraic. Only for motivation, we mention that every closed semialgebraic set can be expressed as a finite union of such sets [PD, 2.4.1]. (Moreover, a deep theorem of Bröcker and Scheiderer says that the number s of inequalities needed to write such a set can be bounded by $n(n+1)/2$ [BCR, 10.4.8].)

In this article, we will be concerned with representations of polynomials that certify nonnegativity on *compact* such sets. We will prove the following theorem:

Theorem 1. *Suppose $p_1, \dots, p_s \in \mathbb{R}[\bar{X}]$ are polynomials such that the semialgebraic set $S := \{p_1 \geq 0, \dots, p_s \geq 0\} \subset \mathbb{R}^n$ they define is compact. Now*

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(a) either set

$$T := \left\{ \sum_{\alpha \in \mathbb{N}^s} \lambda_\alpha p^\alpha \mid \textit{all } \lambda_\alpha \in [0,\infty), \textit{ only finitely many } \neq 0 \right\}$$

and assume that there are linear (i.e., degree ≤ 1) polynomials $l_1, \dots, l_r \in T$ such that the polyhedron $\{l_1 \geq 0, \dots, l_r \geq 0\} \supset S$ is compact

(b) or fix an odd number $d \in \mathbb{N}$ and set

$$T := \left\{ \sum_{\alpha \in \{0,\dots,2d-1\}^s} \sigma_\alpha p^\alpha \mid \sigma_\alpha \textit{ is a sum of } 2d\textit{-th powers in } \mathbb{R}[\bar{X}] \right\}.$$

Suppose $f \geq 0$ on S and there is an identity

$$f = g_1 h_1 + \dots g_m h_m \tag{1}$$

such that $h_i \in T$ and $g_i > 0$ on $S \cap \{f = 0\}$. Then $f \in T$.
(When $f > 0$ on S the required identity always exists, e.g., $f = f \cdot 1$. Moreover, the theorem was already known in this case, see below.)

In both cases (a) and (b), T is defined as a set of polynomials which are nonnegative on S and possess a certain certificate of nonnegativity. The existence of such certificates has recently become an issue in mathematical optimization: Suppose you want to compute numerically the infimum f^* of a polynomial f on a non-empty compact basic closed semialgebraic set S. Equivalently, you can compute the maximal lower bound of f on S, i.e., the maximal $\mu \in \mathbb{R}$ such that $f - \mu \geq 0$ on S. Now suppose we are in case (a). For each fixed k, computing the maximal μ such that $f - \mu = \sum_{|\alpha| \leq k} \lambda_\alpha p^\alpha$ for some $\lambda_\alpha \in [0,\infty)$ ($|\alpha| := \alpha_1 + \dots + \alpha_n \leq k$) amounts to solving a linear program (cf. [L2]). It is a natural question whether for some k, the optimal value of this linear program is exactly f^*. This is the case if and only if $f - f^*$ (a nonnegative polynomial with at least one zero) lies in T. One gets a similar scheme for case (b) when $d = 1$: For each fixed k, the problem of computing the maximal μ such that $f - \mu = \sum_{\alpha \in \{0,1\}^s} \sigma_\alpha p^\alpha$ for some sum of squares σ_α with $\deg(\sigma_\alpha p^\alpha) \leq k$ can be expressed as a semidefinite program (semidefinite programming is a generalization of linear programming). See [L1] or [Sw4].

Example 2. Set $f := X^4Y^2 + X^2Y^4 - 3X^2Y^2 + 1 \in \mathbb{R}[X,Y]$. This is the famous Motzkin polynomial, the first explicitly known example of a polynomial which is nonnegative on $\mathbb{R}^n$ but not a sum of squares of polynomials (see [Rez]). It vanishes (exactly) at the four points $(\pm 1, \pm 1)$. Let the compact set $S = [-1,1]^2$ in Theorem 1 be defined by the $s := 4$ polynomials

$$p_1 := 1 - X, \quad p_2 := 1 + X, \quad p_3 := 1 - Y, \quad p_4 := 1 + Y$$

and define T as in alternative (a) (S itself is a polyhedron as required). Obviously,

$$f = Y^2 p_1^2 p_2^2 + X^2 p_3^2 p_4^2 + p_1 p_2 p_3 p_4 \tag{2}$$

serves as an identity (1) because X^2, Y^2 and the constant polynomial 1 take positive values in the four points $(\pm 1, \pm 1)$. Therefore $f \in T$ by Theorem 1. We were actually able to find such a representation using linear programming:

$$f = \frac{1}{8}(p_1^3p_3^2p_4^2 + p_1^2p_2^2p_3^3 + p_1^2p_2^2p_4^3 + p_2^3p_3^2p_4^2 + p_1^3p_2p_3p_4) + \\ \frac{1}{16}(p_1^2p_2p_3^3p_4 + p_1p_2^3p_3^2p_4 + p_1p_2^3p_3p_4^2 + p_1p_2^2p_3^3p_4 + p_1p_2p_3^2p_4^3 + p_1p_2p_3p_4^4)$$

Note that almost all terms in this sum have degree 7 whereas in (2) all terms had degree ≤ 6. To illustrate the feature that the g_i in (1) are allowed to take negative values outside of $S \cap \{f = 0\}$, note that for any $\lambda \in \mathbb{R}$, the equation

$$f = ((1-\lambda)Y^2 + \lambda)p_1^2p_2^2 + ((1-\lambda)X^2 + \lambda)p_3^2p_4^2 \\ + (\lambda X^2 + \lambda Y^2 + 1 - 2\lambda)p_1p_2p_3p_4$$

(of which (2) is the special case $\lambda = 0$) does the same job as (2).

The deeper reason why in Theorem 1 the choices of $T \subset \mathbb{R}[\bar{X}]$ according to (a) and (b) are possible, is that in both cases

$$S(T) := \{x \in \mathbb{R}^n \mid t(x) \geq 0 \text{ for all } t \in T\}$$

obviously equals S, and T is a weakly divisible archimedean semiring of $\mathbb{R}[\bar{X}]$ (see Definition 3 below). The latter is far from being trivial, at least in case (b):

In case (a), it follows from Proposition 4 below and an old theorem of Minkowski on linear inequalities [PD, 5.4.5] (express $N \pm X_i$ for big $N \in \mathbb{N}$ as a nonnegative linear combination of $1, l_1, \dots, l_r \in T$, confer also [H4]).

In case (b) and $d = 1$, it was first proved by Schmüdgen [Sch] in 1991 combining functional analysis and the Positivstellensatz (a “real” analogue of Hilbert’s Nullstellensatz [PD] [BCR]). For general odd d, it was proved algebraically by Berr and Wörmann [BW].

We will show a version of Theorem 1 where T is an arbitrary weakly divisible archimedean semiring of $\mathbb{R}[\bar{X}]$ and $S = S(T)$. A point x in $\mathbb{R}^n$ can be regarded as a ring homomorphism $\mathbb{R}[\bar{X}] \to \mathbb{R}$ (sending X_i to x_i) and vice versa. Now looking at the elements of an arbitrary commutative ring A as a function on the set of ring homomorphisms $A \to \mathbb{R}$, we will arrive at a more general and abstract version of Theorem 1, namely Theorem 12 in Section 3 below.

In Section 2, we will introduce the abstract framework in which Theorem 12 will be proved. The proof will be carried out in Section 3. In a special case which implies case $d = 1$ in (b) of Theorem 1, the theorem follows almost from recent work of Scheiderer, Kuhlmann, Marshall and Schwartz. This alternative approach will be exposed in Section 4. It does not extend to the general situation we encounter here.

However, after Prof. David Handelman has looked at our preprint, he informed us that (a perhaps insignificantly less general version of) our result can be proved in a completely different function-analytic way using [H3, Theorem I.1] and [H3, Proposition I.2(c)] (which we found to be suitable versions of Eidelheit’s very old separation theorem [Jam, 0.2.4] [Köt, §17.1(3)] and an old result of Bonsall, Lindenstrauss and Phelps [BLP, Theorem 10]). Both approaches are independently of interest. Based on Handelman’s ideas, the author was in the meanwhile able to obtain an extension of a representation theorem of Putinar and Jacobi [Put] [Jac] (again, from positivity to nonnegativity). Therefore, Handelman’s approach will be discussed in a future publication rather than here.

In Section 5, we apply (the more abstract version Theorem 12 of) our criterion to give for the first time a purely ring-theoretic proof of a nice theorem of Handelman saying inter alia the following: If some odd power of a real polynomial in several variables has only nonnegative coefficients, then so do all sufficiently high powers. See Theorem 22 and Corollary 23.

For strict positivity instead of nonnegativity, our criterion Theorem 12 reduces to a classical criterion which is Corollary 14 in this work. It is going back to Krivine, Stone, Kadison et al. It used to be called Kadison-Dubois theorem but due to its (to some extent only recently revealed) complex history (see [PD, Section 5.6]) it is now often called Real Representation Theorem. From this classical criterion and the archimedean property of T (due to Minkowski in (a) and Schmüdgen, Berr and Wörmann in (b), see the discussion above), Theorem 1 was already known for the case $f > 0$ on S.

Scheiderer proved Theorem 1 in the case where $d = 1$ in (b) and equation (1) is of the special form $f = g_1 h_1 + 1 \cdot h_2$ [S3, Proposition 3.10]. Using only this special case, he gave nice geometric criteria for a polynomial which is nonnegative with only finitely many zeros on S to lie in T (T defined as in (b) for $d = 1$). See [S3, Example 3.18] or (for even greater generality) [Mar, Theorem 2.3]).

2. Archimedean semirings

Throughout this article, A denotes a commutative ring. The case where the unique ring homomorphism $\mathbb{Z} \to A$ (all rings have a unit element and all ring homomorphisms preserve unit elements) is not an embedding is formally admitted but our results will be trivial in this case. So the reader might assume that A contains $\mathbb{Z}$ as a subring. Whenever we postulate that $\frac{1}{r}$ lies in A for some integer $r \geq 2$, we implicitly require that r (that is $r \cdot 1$) is a unit of A (i.e., invertible in A).

Definition 3. A set $T \subset A$ is called a *semiring* of A if $0, 1 \in T$ and T is closed under addition and multiplication, i.e., $T + T \subset T$ and $TT \subset T$. A semiring T of A is called a *preorder* of A if it contains all the squares of A, i.e., $A^2 \subset T$. We call a semiring T *archimedean* (with respect to A) if $\mathbb{Z} + T = A$. We call a semiring *weakly divisible* if there is some integer $r \geq 2$ with $\frac{1}{r} \in T$.

Semirings in our sense (i.e., as subsets of rings) are often called *preprimes* (cf. [PD, Definition 5.4.1]). This goes back to Harrison who called these objects *infinite* preprimes (opposing them to his *finite* preprimes) which makes sense in a certain number theoretic context [Har]. However, without the adjective "infinite" and in a general context, this terminology is hermeneutic.

For any semiring $T \subset A$, we set

$$S_A(T) := S(T) := \{\varphi \mid \varphi : A \to \mathbb{R} \text{ ring homomorphism}, \varphi(T) \subset [0, \infty)\}$$

where the topology on $S(T)$ is induced by the product topology on $\mathbb{R}^A$, i.e., is the weakest topology making $S(T) \to \mathbb{R} : \varphi \mapsto \varphi(a)$ for all $a \in A$ continuous. If T is archimedean, then $S(T)$ is easily seen to be compact (meaning quasi-compact and Hausdorff): Choose for each $a \in A$ some $N_a \in \mathbb{N}$ with $N_a \pm a \in T$. Then $S(T)$ is a closed subset of the topological space $\prod_{a \in A} [-N_a, N_a]$ (which is compact by Tychonoff's theorem).

We now have a ring homomorphism

$$A \to \mathcal{C}(S(T), \mathbb{R}) : a \mapsto (\varphi \mapsto \varphi(a))$$

sending all $a \in T$ to a function nonnegative on the whole of $S(T)$. When we write a, we will often mean the image under this map. In this sense, $\varphi(a) = a(x)$ for all $x := \varphi \in S(T)$.

Often, $S(T)$ takes on a very concrete form. For any set $P \subset \mathbb{R}[\bar{X}]$, we define

$$V(P) := \{x \in \mathbb{R}^n \mid p(x) = 0 \text{ for all } p \in P\} \subset \mathbb{R}^n.$$

Suppose that A is finitely generated over a subring R. Then (up to isomorphism) $A = R[\bar{X}]/I$ for some number n of variables and an ideal I of A. If $R \subset \mathbb{R}$ and $[0, \infty) \cap R \subset T$, then every $\varphi \in S(T)$ is the identity on R and it is easy to see that

$$S(T) = \{x \in V(I) \mid t(x) \geq 0 \text{ for all } t \in T\} \subset \mathbb{R}^n \tag{3}$$

via the homeomorphism

$$\varphi \mapsto (\varphi(X_1 + I), \ldots, \varphi(X_n + I)).$$

Even if a semiring $T \subset A$ is not archimedean, there is always a biggest subring $O_T(A) \subset A$ such that $T \cap O_T(A)$ is archimedean This follows from the important Proposition 4 below. So with some additional difficulties (namely determining $S(O_T(A)))$, our membership criterion also gives information about non-archimedean semirings. This will become very clear in Section 5.

Proposition 4. *Let T be a semiring of A. Then*

$$O_T(A) := \{a \in A \mid N \pm a \in T \textit{ for some } N \in \mathbb{N}\}$$

is a subring of A, the ring of T-bounded elements *of A. Moreover, T is archimedean if and only if $O_T(A) = A$.*

Proof. Obviously, $0, 1 \in O_T(A)$ since $0 \pm 0 = 0 \in T$ and $1 \pm 1 \in \{0, 2\} \subset T$. It is immediate from the definition of $O_T(A)$ that $-O_T(A) \subset O_T(A)$. That $O_T(A)$ is closed under addition, follows easily from $T + T \subset T$. To see that it is closed under multiplication, use the two identities

$$3N^2 \mp ab = (N \mp a)(N + b) + N(N \pm a) + N(N - b)$$

and that T is closed under multiplication and addition. We leave the second statement to the reader. □

Without going into details, we make some final remarks on the space $S(T)$. There is a larger topological space one could naturally associate to a semiring T of a ring A, namely the subspace $\operatorname{Sper}_T(A)$ of the so-called real spectrum $\operatorname{Sper}(A)$ of A consisting of all so-called orderings of the ring A lying over T (see, e.g., [PD, 4.1]). Since $S(T) \subset \operatorname{Sper}_T(A)$ via a canonical embedding, all our results will also be true for $\operatorname{Sper}_T(A)$. If T is an archimedean semiring, then $S(T)$ equals $(\operatorname{Sper}_T(A))^{\max}$, the space of maximal orderings of A lying above T. When T is not archimedean, $\operatorname{Sper}_T(A)$ is certainly preferable to $S(T)$ (for example, $\operatorname{Sper}_T(A)$ is even then always quasi-compact). However, we feel that in the context of archimedean semirings we encounter here, the usage of $\operatorname{Sper}_T(A)$ has only disadvantages. For example, unlike $S(T)$, $\operatorname{Sper}_T(A)$ can usually not be really identified with a concrete subset of $\mathbb{R}^n$. Confer also [S3, 2.3].

3. The membership criterion

We begin by introducing some notation. For $\alpha \in \mathbb{N}^n$, we write

$$|\alpha| := \alpha_1 + \cdots + \alpha_n,$$

so that the monomial

$$\bar{X}^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$$

has degree $|\alpha|$. For $x \in \mathbb{R}^n$, $\|x\|$ always denotes the 1-norm of x, i.e.,

$$\|x\| := |x_1| + \cdots + |x_n|.$$

Correspondingly,

$$B_r(x) := \{y \in \mathbb{R}^n \mid \|y - x\| < r\} \qquad (x \in \mathbb{R}^n, 0 < r \in \mathbb{R})$$

denotes the open ball around x of radius r with respect to the 1-norm and

$$\overline{B_r(x)} = \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}$$

its closure. Like all norms, the 1-norm defines the usual topology on $\mathbb{R}^n$. The reason for our choice of this norm is that $\|\alpha\| = |\alpha|$ for $\alpha \in \mathbb{N}^n$. Despite this equality, we want to keep both notations since $|\alpha| = k$ will mean implicitly $\alpha \in \mathbb{N}^n$ (and that α plays the role of a tuple of exponents of a monomial $\bar{X}^\alpha$). We introduce the compact set

$$\begin{aligned}\Delta &:= (\overline{B_1(0)} \setminus B_1(0)) \cap [0,\infty)^n = V(\{X_1 + \cdots + X_n - 1\}) \cap [0,\infty)^n \\ &= \{x \in [0,\infty)^n \mid \|x\| = 1\} \subset \mathbb{R}^n.\end{aligned}$$

For a given set $P \subset \mathbb{R}[\bar{X}]$, we denote by P^+ its subset of all polynomials which have only nonnegative coefficients and by P^* its subset of all homogeneous polynomials (i.e., all of whose nonzero monomials have the same degree).

The starting point for the proof of our criterion is an idea going back to Pólya. Suppose $f \in \mathbb{R}[\bar{X}]^*$. Pólya relates the geometric behaviour of f on the nonnegative orthant $[0,\infty)^n$ with the signs of the coefficients of a “refinement” of f. Due to homogeneity, f can just as well be looked at on Δ instead of $[0,\infty)^n$. Multiplying f by $X_1 + \cdots + X_n$ does not change f on Δ but “refines” the pattern of signs of its coefficients. When we repeat this multiplication sufficiently often, it turns out that the obtained pattern reflects more and more the geometric behaviour of (the sign of f) on $[0,\infty)^n$ (the coefficient of $\bar{X}^\alpha$ in f is more or less related to $f(\alpha)$). The exact statement we will need is formulated in Lemma 5 below. Whereas previous works of the author [Sw1] [Sw3] [Sw4] (see Remark 15 below) required only Pólya's original theorem, we need this time really a more local version where we look at f only on a closed subset U of Δ. Nevertheless, the proof goes exactly along the lines of Pólya (cf. [Pól] [PR]). We include it for the convenience of the reader.

Lemma 5. *Suppose $f \in \mathbb{R}[\bar{X}]^*$ and $U \subset \Delta$ is closed such that $f > 0$ on U. Then there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $0 \neq \alpha \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in U$, the coefficient of $\bar{X}^\alpha$ in $(X_1 + \cdots + X_n)^k f$ is nonnegative.*

Proof. Set $d := \deg f$ and assume without loss of generality $d > 0$. Write $f = \sum_{|\beta|=d} a_\beta \bar{X}^\beta$, $a_\beta \in \mathbb{R}$. We know that

$$(X_1 + \cdots + X_n)^k = \sum_{|\gamma|=k} \frac{k!}{\gamma_1! \cdots \gamma_n!} \bar{X}^\gamma$$

for $k \in \mathbb{N}$. For degree reasons, the coefficient of $\bar{X}^\alpha$ in $(X_1 + \cdots + X_n)^k f$ is of course zero when $|\alpha| \neq k + d$. Now for any $\alpha \in \mathbb{N}^n$ with $|\alpha| = k + d$, the coefficient of $\bar{X}^\alpha$ in $(X_1 + \cdots + X_n)^k f$ equals

$$\begin{aligned}
\sum_{\substack{|\beta|=d,|\gamma|=k\\ \beta+\gamma=\alpha}} \frac{k!}{\gamma_1! \cdots \gamma_n!} a_\beta &= \sum_{\substack{|\beta|=d,|\gamma|=k\\ \beta+\gamma=\alpha}} \frac{k!}{(\alpha_1-\beta_1)! \cdots (\alpha_n-\beta_n)!} a_\beta \\
&= \sum_{\substack{|\beta|=d\\ \beta\leq\alpha}} \frac{k!}{(\alpha_1-\beta_1)! \cdots (\alpha_n-\beta_n)!} a_\beta \\
&= \frac{k!(k+d)^d}{\alpha_1! \cdots \alpha_n!} \sum_{\substack{|\beta|=d\\ \beta\leq\alpha}} a_\beta \prod_{i=1}^n \frac{\alpha_i!}{(\alpha_i-\beta_i)!(k+d)^{\beta_i}} \\
&= \frac{k!(k+d)^d}{\alpha_1! \cdots \alpha_n!} \sum_{|\beta|=d} a_\beta \prod_{i=1}^n \left(\frac{\alpha_i}{k+d}\right)^{\beta_i}_{\frac{1}{k+d}}
\end{aligned}$$

where we abbreviate

$$(a)_b^m := \prod_{i=0}^{m-1} (a - ib).$$

Note that $(a)_0^m = a^m$ to understand the idea behind the notation $(a)_b^m$. Also note that the condition $\beta \leq \alpha$ (meaning $\beta_i \leq \alpha_i$ for all i) has been dropped in the index of summation in the last expression. This is justified since all the corresponding additional terms in the sum are zero. Now we see that the coefficient of $\bar{X}^\alpha$ with $|\alpha| = k + d$ equals up to a positive factor

$$f_{\frac{1}{k+d}}\left(\frac{\alpha}{k+d}\right)$$

where we define

$$f_\varepsilon := \sum_{|\beta|=d} a_\beta (X_1)_\varepsilon^{\beta_1} \cdots (X_n)_\varepsilon^{\beta_n} \in \mathbb{R}[\bar{X}]$$

for all $\varepsilon \in [0, \infty)$. Obviously, f_ε converges to $f_0 = f$ uniformly on U when $\varepsilon \to 0$. Since U is compact and $f > 0$ on U, there is $k_0 \in \mathbb{N}$ such that $f_{\frac{1}{k+d}} > 0$ on U for all $k \geq k_0$, in particular

$$f_{\frac{1}{k+d}}\left(\frac{\alpha}{k+d}\right) > 0$$

whenever $\alpha \in \mathbb{N}^n$ with $|\alpha| = k + d$ and $\frac{\alpha}{k+d} \in U$. □

From this we deduce Pólya's theorem as a corollary, although this will not be used later on. Alternatively, Pólya's theorem follows by taking independently of $x \in \Delta$ the same identity $f = f \cdot 1$ in condition (a) of Lemma 7.

Corollary 6 (Pólya)**.** *Suppose $f \in \mathbb{R}[\bar{X}]^*$ and $f > 0$ on Δ. Then*

$$(X_1 + \cdots + X_n)^k f \in \mathbb{R}[\bar{X}]^+$$

for large $k \in \mathbb{N}$.

Proof. Set $U = \Delta$ in Lemma 5. □

The next lemma reminds already a bit of Theorem 12 below. But note that the g_i and h_i are allowed to depend on x. The idea is to apply Pólya's refinement process locally on the g_i while the h_i do not disturb too much. Note that we do no longer assume that f is homogeneous. Also observe that the hypotheses imply $f \geq 0$ on Δ.

Lemma 7. *Let $f \in \mathbb{R}[\bar{X}]$. Suppose that for every $x \in \Delta$ there are $m \in \mathbb{N}$, $g_1, \dots, g_m \in \mathbb{R}[\bar{X}]^*$ and $h_1, \dots, h_m \in \mathbb{R}[\bar{X}]^+$ such that*
(a) $f = g_1 h_1 + \cdots + g_m h_m$ and
(b) $g_1(x) > 0, \dots, g_m(x) > 0$.
Then there exists $k \in \mathbb{N}$ such that $(X_1 + \cdots + X_n)^k f \in \mathbb{R}[\bar{X}]^+$.

Proof. Choose a family $(\varepsilon_x)_{x \in \Delta}$ of real numbers $\varepsilon_x > 0$ such that for every $x \in \Delta$, there are $m \in \mathbb{N}$, $g_1, \dots, g_m \in \mathbb{R}[\bar{X}]^*$ and $h_1, \dots, h_m \in \mathbb{R}[\bar{X}]^+$ satisfying (a) and not only (b) but even

$$g_i > 0 \text{ on } \overline{B_{2\varepsilon_x}(x)} \cap \Delta \qquad \text{for } i \in \{1, \dots, m\}. \tag{4}$$

The family $(B_{\varepsilon_x}(x))_{x \in \Delta}$ is an open covering of Δ. Since Δ is compact, there is a finite subcovering, i.e., a finite set $D \subset \Delta$ for which $\Delta \subset \bigcup_{x \in D} B_{\varepsilon_x}(x)$, in particular

$$\Delta = \bigcup_{x \in D} (\overline{B_{\varepsilon_x}(x)} \cap \Delta).$$

As D is finite, it suffices to show for fixed $x \in D$, that there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and all $0 \neq \alpha \in \mathbb{N}^n$ with

$$\frac{\alpha}{|\alpha|} \in \overline{B_{\varepsilon_x}(x)}, \tag{5}$$

the coefficient of $\bar{X}^\alpha$ in $(X_1 + \cdots + X_n)^k f$ is nonnegative.

Therefore fix $x \in D$. By choice of ε_x, we find $m \in \mathbb{N}$, $g_1, \dots, g_m \in \mathbb{R}[\bar{X}]^*$ and $h_1, \dots, h_m \in \mathbb{R}[\bar{X}]^+$ satisfying (a) and (4). For every $i \in \{1, \dots, m\}$, the positivity condition (4) enables us to apply Lemma 5 to g_i, yielding $k_i \in \mathbb{N}$ such that for all $k \geq k_i$ and all $0 \neq \beta \in \mathbb{N}^n$ with

$$\frac{\beta}{|\beta|} \in \overline{B_{2\varepsilon_x}(x)}, \tag{6}$$

the coefficient of $\bar{X}^\beta$ in $(X_1 + \cdots + X_n)^k g_i$ is nonnegative (use that $\frac{\beta}{|\beta|} \in \Delta$ is automatic). Choose moreover $1 \leq l \in \mathbb{N}$ so large that

$$\frac{2|\gamma|}{l} \leq \varepsilon_x \tag{7}$$

for all $\gamma \in \mathbb{N}^n$ for which the coefficient of $\bar{X}^\gamma$ in at least one of the polynomials $h_1, \dots, h_m$ does not vanish. Set

$$k_0 := \max\{k_1, \dots, k_n, l\}.$$

Let $k \geq k_0$ and suppose $0 \neq \alpha \in \mathbb{N}^n$ satisfies (5). Fix $i \in \{1, \dots, m\}$. By equation (a), it is enough to show that the coefficient of $\bar{X}^\alpha$ in $(X_1 + \cdots + X_n)^k g_i h_i$ is nonnegative. This coefficient is of course a sum of certain products of coefficients of $(X_1 + \cdots + X_n)^k g_i$ and h_i. But all the concerned products are nonnegative. Indeed, consider $\beta, \gamma \in \mathbb{N}^n$ with $\beta + \gamma = \alpha$ (i.e., $\bar{X}^\beta \bar{X}^\gamma = \bar{X}^\alpha$) such that the corresponding coefficients of $\bar{X}^\beta$ in $(X_1 + \cdots + X_n)^k g_i$ and $\bar{X}^\gamma$ in h_i do not vanish. The latter coefficient is positive since $h_i \in \mathbb{R}[\bar{X}]^+$. We show that the other one is positive, too. From degree consideration it is trivial that $|\beta| \geq k \geq k_0 \geq l \geq 1$ which implies together with the now satisfied condition (7)

$$\frac{2|\gamma|}{|\beta|} \leq \varepsilon_x. \tag{8}$$

We exploit this to verify condition (6) which is all we need since $k \geq k_0 \geq k_i$:

$$\begin{aligned}\left\|\frac{\beta}{|\beta|} - x\right\| &\leq \left\|\frac{\beta}{|\beta|} - \frac{\alpha}{|\alpha|}\right\| + \left\|\frac{\alpha}{|\alpha|} - x\right\| \overset{(5)}{\leq} \varepsilon_x + \left\|\frac{|\alpha|\beta - |\beta|\alpha}{|\alpha||\beta|}\right\| \\ &= \varepsilon_x + \frac{\||\alpha|\gamma - |\gamma|\alpha\|}{|\alpha||\beta|} \leq \varepsilon_x + \frac{\||\alpha|\gamma\| + \||\gamma|\alpha\|}{|\alpha||\beta|} \\ &= \varepsilon_x + \frac{2|\alpha||\gamma|}{|\alpha||\beta|} = \varepsilon_x + \frac{2|\gamma|}{|\beta|} \overset{(8)}{\leq} 2\varepsilon_x\end{aligned}$$

□

Now we deal with the case where the g_i are no longer assumed to be homogeneous.

Lemma 8. *Let $f \in \mathbb{Z}[\bar{X}]$ such that for all $x \in \Delta$, there exist $m \in \mathbb{N}$, $g_1, \dots, g_m \in \mathbb{Z}[\bar{X}]$ and $h_1, \dots, h_m \in \mathbb{Z}[\bar{X}]^+$ such that*

(a) $f = g_1h_1 + \dots + g_mh_m$ and

(b) $g_1(x) > 0, \dots, g_m(x) > 0$.

Then f is modulo the principal ideal $\mathbb{Z}[\bar{X}](X_1 + \dots + X_n - 1)$ congruent to a polynomial without negative coefficients.

Proof. For every $x \in \Delta$, choose $m_x \in \mathbb{N}$, $g_{x1}, \dots, g_{xm_x} \in \mathbb{Z}[\bar{X}]$ and $0 \neq h_{x1}, \dots, h_{xm_x} \in \mathbb{Z}[\bar{X}]^+$ according to (a) and (b). Setting

$$U_x := \{y \in \Delta \mid g_{x1}(y) > 0, \dots, g_{xm_x}(y) > 0\}, \tag{9}$$

we have $x \in U_x$ for $x \in \Delta$. Therefore $(U_x)_{x\in\Delta}$ is an open covering of the compact set Δ and possesses a finite subcovering, i.e., there is a finite set $D \subset \Delta$ such that

$$\Delta = \bigcup_{x\in D} U_x. \tag{10}$$

Choose an upper bound $d \in \mathbb{N}$ for the degrees of the (in each case m_x) terms appearing in the sums on the right hand sides of the equations (a) corresponding to the finitely many $x \in D$, i.e.,

$$d \geq \deg g_{xi} + \deg h_{xi} \qquad \text{for all } x \in D \text{ and } i \in \{1, \dots, m_x\}.$$

Fix for the moment such a pair (x, i) and choose $d', d'' \in \mathbb{N}$ such that $d = d' + d''$, $d' \geq \deg g_{xi}$ and $d'' \geq \deg h_{xi}$. Write $g_{xi} = \sum_{k=0}^{d'} p_k$ and $h_{xi} = \sum_{k=0}^{d''} q_k$ where $p_k, q_k \in \mathbb{Z}[\bar{X}]$ are homogeneous of degree k (if not zero). Set

$$g'_{xi} := \sum_{k=0}^{d'} (X_1 + \dots + X_n)^{d'-k} p_k \quad \text{and} \qquad h'_{xi} := \sum_{k=0}^{d''} (X_1 + \dots + X_n)^{d''-k} q_k.$$

Now g'_{xi} and h'_{xi} are homogeneous polynomials whose product is (homogeneous) of degree d (if not zero). Then $g'_{xi} \equiv g_{xi}$ and $h'_{xi} \equiv h_{xi}$ modulo $\mathbb{Z}[\bar{X}](X_1+\dots+X_n-1)$, in particular, g'_{xi} coincides with g_{xi} on Δ. Moreover, h'_{xi} inherits the property of having no negative coefficients from h_{xi}. For every $x \in D$,

$$f'_x := g'_{x1}h'_{x1} + \cdots + g'_{xm_x}h'_{xm_x} \in \mathbb{Z}[\bar{X}]^* \tag{11}$$

is homogeneous of degree d (unless zero) and congruent to f modulo $\mathbb{Z}[\bar{X}](X_1 + \cdots + X_n - 1)$. For $x, y \in D$, $f'_x - f'_y$ is therefore homogeneous and at the same time a multiple of $X_1 + \cdots + X_n - 1$. Hence actually $f'_x = f'_y$, i.e., there is $f' \in \mathbb{Z}[\bar{X}]$ such that $f' = f'_x$ for all $x \in D$ and $f' \equiv f$ modulo $\mathbb{Z}[\bar{X}](X_1 + \cdots + X_n - 1)$.

We want to apply Lemma 7 to f'. The hypotheses are now rather easy to verify: Let $x \in \Delta$. By (10), we find $x \in D$ such that $x \in U_x$. Set $m := m_x$, $g_i := g'_{xi}$ and $h_i := h'_{xi}$ for $i \in \{1, \dots, m\}$. Then equation (11) becomes condition (a) in Lemma 7 (with f' instead of f). To verify (b) of Lemma 7, use that $g_i = g'_{xi}$ equals g_{xi} on Δ which is positive in $x \in U_x \subset \Delta$ by (9). By Lemma 7, we get therefore $k \in \mathbb{N}$ such that $(X_1 + \cdots + X_n)^k f'$ has no negative coefficients. But this polynomial is congruent to f' which is in turn congruent to f modulo $\mathbb{Z}[\bar{X}](X_1+\cdots+X_n-1)$. □

To understand the next lemma, assume first that we are in the case where I is the principal ideal of $\mathbb{Z}[\bar{X}]$ generated by $X_1 + \cdots + X_n - 1$. Observing that $V(I) \cap [0, \infty)^n = \Delta$ and the identity (a) in the previous lemma can taken to be $f = f \cdot 1$ at all points of Δ where f is positive, this lemma then is an immediate consequence of the preceding one. Now to get the lemma for a general ideal I, we give up the feature that the h_i are allowed to depend on x.

Lemma 9. *Let I be an ideal of $\mathbb{Z}[\bar{X}]$ such that $X_1 + \cdots + X_n - 1 \in I$. Suppose $m \in \mathbb{N}$, $f \in \mathbb{Z}[\bar{X}]$ and $h_1, \dots, h_m \in \mathbb{Z}[\bar{X}]^+$ such that*
(a) $f \geq 0$ on $V(I) \cap [0, \infty)^n$ and
(b) for all $x \in V(I \cup \{f\}) \cap [0, \infty)^n$, there exist $g_1, \dots, g_m \in \mathbb{Z}[\bar{X}]$ such that
   (i) $f = g_1h_1 + \cdots + g_mh_m$ and
   (ii) $g_1(x) > 0, \dots, g_m(x) > 0$.
Then f is modulo I congruent to a polynomial without negative coefficients.

Proof. Set $U := \{x \in \Delta \mid f(x) > 0\}$ and introduce the set $W \subset \Delta$ of all $x \in \Delta$ for which there are $g_1, \dots, g_m$ fulfilling (i) and (ii). The sets U and W are open in Δ and

$$V(I) \cap [0, \infty)^n \subset U \cup W \tag{12}$$

by (a) and (b). By Hilbert's Basis Theorem, every ideal of $\mathbb{Z}[\bar{X}]$ is finitely generated. In particular, we find $s \in \mathbb{N}$ and $p_1, \dots, p_s \in \mathbb{Z}[\bar{X}]$ such that

$$I = \mathbb{Z}[\bar{X}]p_1 + \cdots + \mathbb{Z}[\bar{X}]p_s + \mathbb{Z}[\bar{X}](X_1 + \cdots + X_n - 1).$$

Setting $p := \sum_{i=1}^s p_i^2 \in I$, we have $p \in I$, $p \geq 0$ on $\mathbb{R}^n$ and

$$p > \varepsilon \text{ on } \Delta \setminus (U \cup W) \qquad \text{for some } \varepsilon > 0. \tag{13}$$

The latter follows from $p > 0$ on $\Delta \setminus V(I)$, (12) and the compactness of $\Delta \setminus (U \cup W)$.

Now we distinguish two cases. First case: $W = \emptyset$. From (13) and the boundedness of f on the compact set $\Delta \setminus U$, we get $k \in \mathbb{N}$ such that $f' := f + kp > 0$ on $\Delta \setminus U$. On the other hand, $f' = f + kp \geq f > 0$ on U. Altogether we get $f' > 0$ on Δ. Now we can clearly apply Lemma 8 to f'. In fact, for every $x \in \Delta$, $f' = f' \cdot 1$ serves as an identity as required in (a) of that lemma. Hence that lemma yields that f' is congruent to a polynomial without negative coefficients modulo $\mathbb{Z}[\bar{X}](X_1 + \cdots + X_n - 1) \subset I$. But $f \equiv f + kp = f'$ modulo I.

Second case: $W \neq \emptyset$. All we really use from $W \neq \emptyset$ is that $f \in \mathbb{Z}[\bar{X}]h_1 + \cdots + \mathbb{Z}[\bar{X}]h_m$ by (i), i.e., we find $q_1, \ldots, q_m \in \mathbb{Z}[\bar{X}]$ such that

$$f = q_1 h_1 + \cdots + q_m h_m. \tag{14}$$

From (13) and the boundedness of $q_1, \ldots, q_m$ on the compact set $\Delta \setminus (U \cup W)$, it follows that we can choose $k \in \mathbb{N}$ such that

$$g_i^{(0)} := q_i + kp > 0 \qquad \text{on } \Delta \setminus (U \cup W) \text{ for all } i \in \{1, \ldots, m\}. \tag{15}$$

We will apply Lemma 8 to

$$f' := g_1^{(0)} h_1 + \cdots + g_m^{(0)} h_m. \tag{16}$$

Note that

$$f' \overset{(15)}{=} \underbrace{q_1 h_1 + \cdots + q_m h_m}_{= f \text{ by } (14)} + kp(\underbrace{h_1 + \cdots + h_m}_{\geq 0 \text{ on } [0,\infty)^n}) \geq f \qquad \text{on } [0, \infty)^n. \tag{17}$$

To check its applicability, let $x \in \Delta$. We consider three different subcases:

First, consider the case where $x \in U$. Then $f'(x) \geq f(x) > 0$ and

$$f' = f' \cdot 1 \tag{18}$$

is an identity as demanded in (a) of Lemma 8.

Second, suppose $x \in W$. By definition of W, we can choose $g_1, \ldots, g_m \in \mathbb{Z}[\bar{X}]$ satisfying (i) and (ii). Set $g_i' := g_i + kp$ for $i \in \{1, \ldots, m\}$. Then

$$\begin{aligned} f' &\overset{(17)}{=} f + kp(h_1 + \cdots + h_m) \\ &\overset{(i)}{=} g_1 h_1 + \cdots + g_m h_m + kp(h_1 + \cdots + h_m) = g_1' h_1 + \cdots + g_m' h_m \end{aligned} \tag{19}$$

serves as a relation as required in (a) of Lemma 8. Note that

$$g_i'(x) = g_i(x) + kp(x) \geq g_i(x) \overset{(ii)}{>} 0 \qquad \text{for } i \in \{1, \ldots, m\}.$$

Third and last, for all $x \in \Delta \setminus (U \cup W)$, (15) allows us to use one and the same equation for (a) of Lemma 8, namely (16).

All in all, Lemma 8 applies now to f', i.e., f' is congruent to a polynomial without nonnegative coefficients modulo $\mathbb{Z}[\bar{X}](X_1 + \cdots + X_n - 1) \subset I$. But $f \equiv f + kp = f'$ modulo I. □

Remark 10. In Lemma 8, the h_i are permitted to depend on x. In the proof of Lemma 9, we do not exploit this too much. Indeed, two of all three used identities (16), (18) and (19) are based on the same h_i, and the third one is trivial.

Proposition 11. *For all weakly divisible semirings T of A are equivalent:*

(i) T is finitely generated and archimedean with respect to $\mathbb{Z}[T]$.
(ii) T is generated by finitely many $y_1, \ldots, y_n \in T$ with $y_1 + \cdots + y_n \in \mathbb{N}$.
(iii) T is generated by finitely many $y_1, \ldots, y_n \in T$ with $y_1 + \cdots + y_n = 1$.

Proof. Easy. Use the weak divisibility for the implication (ii) $\Longrightarrow$ (iii) and Proposition 4 for (iii) $\Longrightarrow$ (i). □

Let us use abbreviations like $S(T) \cap \{a = 0\} := \{x \in S(T) \mid a(x) = 0\}$. Now we attack the main theorem. Note that its hypotheses imply

$$S(T) \cap \{a = 0\} = S(T) \cap \{t_1 = 0, \ldots, t_m = 0\}.$$

Theorem 12. *Let T be a weakly divisible archimedean semiring of A and $a \in A$. Suppose $a \geq 0$ on $S(T)$ and there is an identity*

$$a = b_1 t_1 + \dots + b_m t_m$$

with $b_i \in A$, $t_i \in T$ such that $b_i > 0$ on $S(T) \cap \{a = 0\}$ for all $i \in \{1, \dots, m\}$. Then $a \in T$.

Proof. If the ring homomorphism $\mathbb{Z} \to A$ is not injective, then $-1 \in T$ whence $T = \mathbb{Z} + T = A$. Therefore we assume from now on that A contains $\mathbb{Z}[\frac{1}{r}]$ as a subring and $\frac{1}{r} \in T$ for some integer $r \geq 2$.

Let $\mathcal{I}$ be the set of all finitely generated semirings $T' \subset T$ with $\frac{1}{r}, t_i \in T'$ and $a, b_i \in \mathbb{Z}[T']$ which are archimedean *with respect to* $\mathbb{Z}[T']$. Using (ii) of Proposition 11, we see that $\mathcal{I}$ is a directed partially ordered set (with respect to set inclusion) and that the union over all $T' \in \mathcal{I}$ is T. For any $T' \in \mathcal{I}$, consider the compact (since T' is archimedean with respect to $\mathbb{Z}[T']$) topological space

$$X(T') := \left\{x \in S_{\mathbb{Z}[T']}(T') \mid a(x) \leq 0\right\} \cup \bigcup_{i=1}^{m} \left\{x \in S_{\mathbb{Z}[T']}(T') \mid b_i(x) \leq 0\right\}.$$

Note that we understand T' here as a semiring of the ring $\mathbb{Z}[T'] \subset A$ it generates. So the elements of $X(T')$ are ring homomorphisms defined on the ring $\mathbb{Z}[T']$. Now for all $T', T'' \in I$ with $T' \subset T''$, we have the natural restriction map $X(T'') \to X(T')$ which is continuous. This defines an inverse system in the category of topological spaces indexed by $\mathcal{I}$. Every element of its inverse limit would give rise to a real valued map defined on $\mathbb{Z}[T] = A$ (note that $A = \mathbb{Z} + T$ since T is archimedean) which would even be a ring homomorphism $\varphi : A \to \mathbb{R}$ with $\varphi(T) \subset [0, \infty)$ satisfying $\varphi(a) \leq 0$ or $\varphi(b_i) \leq 0$ for some i. By hypothesis, such a homomorphism cannot exist. Because an inverse limit of non-empty compact spaces is not empty (recall that compact includes Hausdorff), we get henceforth that $X(T') = \emptyset$ for some $T' \in \mathcal{I}$. This means that the hypotheses of the theorem are satisfied for $(\mathbb{Z}[T'], T')$ instead of (A, T).

Therefore, we can assume from now on that T is finitely generated, say by the elements of a tuple $\bar{y} = (y_1, \dots, y_n)$ and consequently $A = \mathbb{Z} + T = \mathbb{Z}[\bar{y}]$. By (iii) of Proposition 11, we can even assume

$$y_1 + \dots + y_n = 1. \tag{20}$$

Now consider the ring epimorphism $\mathbb{Z}[\bar{X}] \to \mathbb{Z}[\bar{y}]$ mapping X_i to y_i for every $i \in \{1, \dots, n\}$. Calling its kernel I, it induces a ring isomorphism $\mathbb{Z}[\bar{X}]/I \to A$ mapping $X_i + I$ to y_i. Without loss of generality, we may assume

$$A = \mathbb{Z}[\bar{X}]/I \qquad \text{and} \qquad y_i = X_i + I \text{ for } i \in \{1, \dots, n\}. \tag{21}$$

Then it follows from (21) that

$$T = \{p + I \mid p \in \mathbb{Z}[\bar{X}]^+\}. \tag{22}$$

and therefore

$$S(T) = V(I) \cap [0, \infty)^n. \tag{23}$$

Choose $g_1, \dots, g_m \in \mathbb{Z}[\bar{X}]$ and $h_1, \dots, h_m \in \mathbb{Z}[\bar{X}]^+$ such that $b_i = g_i + I$, $t_i = h_i + I$ for all i (use (22)). Now set

$$f := g_1 h_1 + \dots + g_m h_m \in \mathbb{Z}[\bar{X}] \tag{24}$$

which is nothing else than condition (i) in Lemma 9. The remaining hypotheses of Lemma 9 are now provided by (20), (21) and (23). That lemma yields that f is congruent to a polynomial without negative coefficients modulo I. By (22), this means that $a = f + I \in T$. □

Together with Remark 10, the next remark will tell us that the intermediate results in this section have not been exploited to their full extent. This gives hope that the just proved theorem can still be improved at least in certain special situations.

Remark 13. In condition (b) of Lemma 9, the g_i are allowed to depend on x. When we apply this lemma in Theorem 12, we do not make use of this.

As a corollary we get the classical result of Stone, Kadison, Krivine et al. (see [PD, Section 5.6] and confer introduction).

Corollary 14 (Real Representation Theorem)**.** *Let T be a weakly divisible archimedean semiring of A. Suppose that $a \in A$ satisfies $a > 0$ on $S(T)$. Then $a \in T$.*

Proof. Use $a = a \cdot 1$ as the required identity in the previous theorem. □

Remark 15. It is instructive to see how one could simplify the argument in this section when one is content with proving Corollary 14 rather than extending it to Theorem 12. The whole section then reduces to what is essentially already contained in the author's earlier work [Sw1] (see also [Sw3]).

4. Alternative proof for preorders

In this section, we demonstrate that Theorem 12 can easily be deduced from recent work of Scheiderer, Kuhlmann, Marshall and Schwartz but only in the case where T is a preorder. The following key lemma and its proof is essentially [KMS, Corollary 2.2].

Lemma 16 (Kuhlmann, Marshall, Schwartz)**.** *Let T be an archimedean preorder of A. Suppose $1 \in Aa + Ab$, $a, b \geq 0$ on $S(T)$ and $ab \in T$. Then $a, b \in T$.*

Proof. By our hypothesis and [KMS, Lemma 2.1] (see also [S3, Proposition 2.7] or [Mar, Lemma 3.2] for a natural generalization of this not needed here), we have $s, t \in A$ such that $1 = sa + tb$ and $s, t > 0$ on $S(T)$. By the classical Real Representation Theorem 14, we have $s, t \in T$. Now $a = sa^2 + tab \in T$ (here we use that $A^2 \subset T$). Symmetrically, we have of course $b \in T$. □

The next example shows that this key lemma does no longer hold in the general situation where T is only assumed to be a semiring instead of a preorder.

Example 17. Let $A := \mathbb{R}[X]$ and $T \subset A$ be the semiring generated by $[0, \infty)$ and the three polynomials $1 \pm X$ and $X^2 + X^4$. The elements of T are the nonnegative linear combinations of products of these polynomials. By Proposition 4, T is clearly archimedean. Setting $a := X^2$ and $b := 1 + X^2$, we clearly have $1 \in Aa + Ab$ and $ab \in T$. Being sums of squares, a and b are of course nonnegative on $S(T)$. We claim that $a \notin T$. Otherwise, we would have an identity

$$X^2 = \sum_{\alpha \in \mathbb{N}^3} \lambda_\alpha (1 + X)^{\alpha_1} (1 - X)^{\alpha_2} (X^2 + X^4)^{\alpha_3} \qquad (\lambda_\alpha \geq 0).$$

Evaluating at 0, we would get that the sum over all λ_α with $\alpha_3 = 0$ is 0. But then, those λ_α would have to equal zero since they are nonnegative. As a consequence, $X^2 + X^4$ would divide X^2 which is absurd.

The idea for the next proof is from Corollaries 2.3 and 2.4 in [KMS].

Alternative proof of Theorem 12 in case $A^2 \subset T$. The set

$$T' := T - a^2 T \subset A$$

is an archimedean preorder and we have $S(T) \cap \{a = 0\} = S(T')$. By hypothesis, we have therefore $b_i > 0$ on $S(T')$ for all i. From the classical Real Representation Theorem 14, we obtain $b_i \in T'$ for all i. Regarding the identity from the hypotheses, this entails $a \in T'$, i.e., $a(1 + at) \in T$ for some $t \in T$. By Lemma 16, therefore $a \in T$. □

Even if Lemma 16 were true for semirings instead of preorders (which is not the case), this alternative proof would break down. We would have to replace the preordering T' generated by T and $-a^2$ by the semiring $T - a^2T + a^4T - a^6T + \dots$ generated by T and $-a^2$. But then we would get only that

$$a(1 + at_1 - a^3t_3 + a^5t_5 - a^7t_7 + \dots) \in T \qquad \text{for some } t_1, t_3, \dots \in T$$

instead of $a(1 + at) \in T$ for some $t \in T$. The negative signs appearing in the second factor of this product now prevent us from applying Lemma 16.

5. Handelman's Theorem on powers of polynomials

In this section, we show that Theorem 12 can be used to give a new proof of a nice theorem of Handelman on powers of polynomials. See Theorem 22 and Corollary 23 below. The original proof in [H5] relies on some nontrivial facts from a whole theory of a certain class of partially ordered abelian groups which is to a large extent due to Handelman. Some of the used facts would not make sense in our ring-theoretic setting, e.g., [H3, Proposition I.2(c)]. Though a lot of ideas are borrowed from Handelman's original work [H1] [H5] (see also [AT]), the proof of Theorem 22 differs considerably from Handelman's original argumentation. Also, we stress a new valuation theoretic viewpoint. We will however only use the most basic facts and notions from valuation theory as they can be found, for example, in the appendix of [PD].

At first glance, it seems that Theorem 12 says nothing about the semiring $\mathbb{R}[\bar{X}]^+$ of $\mathbb{R}[\bar{X}]$. Indeed, $\mathbb{R}[\bar{X}]^+$ is not an archimedean semiring of $\mathbb{R}[\bar{X}]$. However, for a semiring T of a ring A, $T \cap O_T(A)$ is an archimedean semiring of the ring of T-bounded elements $O_T(A) \subset A$ (cf. Proposition 4). Still, this does not seem to help since $O_{\mathbb{R}[\bar{X}]^+}(\mathbb{R}[\bar{X}]) = \mathbb{R}$. When a ring of bounded elements is too small, it is often a good idea to localize it by a fixed element, i.e., to build a new ring where division by this element is allowed (see, e.g., [Sw2, Theorem 5.1] or [PV]). Following Handelman (see, e.g., [H3, p. 61]), we will localize by a fixed $0 \neq g \in \mathbb{R}[\bar{X}]^+$. Hence we consider the ring

$$\mathbb{R}[\bar{X}]_g := \mathbb{R}\left[\bar{X}, \frac{1}{g}\right] = \left\{\frac{f}{g^k} \mid f \in \mathbb{R}[\bar{X}], k \in \mathbb{N}\right\} \subset \mathbb{R}(\bar{X})$$

($\mathbb{R}(\bar{X})$ denoting the quotient field of $\mathbb{R}[\bar{X}]$) together with the semiring

$$T_g := \left\langle T \cup \left\{\frac{1}{g}\right\}\right\rangle = \left\{\frac{f}{g^k} \mid f \in \mathbb{R}[\bar{X}]^+, k \in \mathbb{N}\right\} \subset \mathbb{R}[\bar{X}]_g$$

(we write angular brackets for the generated semiring). For a polynomial $p \in \mathbb{R}[\bar{X}]$, we denote by $\operatorname{Log}(p) \subseteq \mathbb{N}^n$ the set of all $\alpha \in \mathbb{N}^n$ for which the coefficient of $\bar{X}^\alpha$ in p does not vanish. Its convex hull $\operatorname{New}(p) \subset \mathbb{R}^n$ is called the *Newton polytope* of p. It is easy to see that

$$\operatorname{Log}(pq) \subset \operatorname{Log}(p) + \operatorname{Log}(q) \qquad \text{for all } p, q \in \mathbb{R}[\bar{X}], \tag{25}$$

$$\operatorname{Log}(pq) = \operatorname{Log}(p) + \operatorname{Log}(q) \qquad \text{for all } p, q \in \mathbb{R}[\bar{X}]^+ \text{ and} \tag{26}$$

$$\operatorname{New}(pq) = \operatorname{New}(p) + \operatorname{New}(q) \qquad \text{for all } p, q \in \mathbb{R}[\bar{X}]. \tag{27}$$

These basic facts will frequently be used in the sequel, most often tacitly. We now determine the ring of T_g-bounded elements $A(g)$ and its (by Proposition 4) archimedean semiring $T(g) := T_g \cap A_g$:

$$A(g) := O_{T_g}(A_g) = \left\{\frac{f}{g^k} \mid f \in \mathbb{R}[\bar{X}], k \in \mathbb{N}, \operatorname{Log}(f) \subset \operatorname{Log}(g^k)\right\} \tag{28}$$

$$T(g) := T_g \cap A(g) = \left\{\frac{f}{g^k} \mid f \in \mathbb{R}[\bar{X}]^+, k \in \mathbb{N}, \operatorname{Log}(f) \subset \operatorname{Log}(g^k)\right\} \tag{29}$$

The inclusions from right to left are trivial whereas the inclusion from left to right in (28) uses (25) and the one in (29) uses (25) and (26). Using (25), the following becomes clear quickly:

$$A(g) = \mathbb{R}\left[\frac{\bar{X}^\alpha}{g} \mid \alpha \in \operatorname{Log}(g)\right] \tag{30}$$

$$T(g) = \left\langle [0,\infty) \cup \left\{\frac{\bar{X}^\alpha}{g} \mid \alpha \in \operatorname{Log}(g)\right\}\right\rangle \tag{31}$$

Fix an arbitrary $w \in \mathbb{R}^n$. There is exactly one valuation $v_w : \mathbb{R}(\bar{X}) \to \mathbb{R} \cup \{\infty\}$ satisfying

$$v_w(p) = -\max\{\langle w, \alpha\rangle \mid \alpha \in \operatorname{Log}(p)\} \qquad (0 \neq p \in \mathbb{R}[\bar{X}]). \tag{32}$$

This is easy to show by noting that $\operatorname{Log}(p)$ can be replaced by $\operatorname{New}(p)$ in (32) and using (27). Here and elsewhere $\langle w, \alpha\rangle$ denotes the usual scalar product of w and α. We define the *w-initial part* $\operatorname{in}_w(p) \in \mathbb{R}[\bar{X}]$ of a polynomial $p \in \mathbb{R}[\bar{X}]$ as the sum of those monomials appearing in p belonging to an exponent tuple $\alpha \in \mathbb{N}^n$ for which $\langle w, \alpha\rangle$ gets maximal (i.e., equals $-v_w(p)$). The following is easy to check:

$$\operatorname{in}_w(p)(x) = \lim_{t\to\infty} e^{tv_w(p)} p(e^{tw_1}x_1, \ldots, e^{tw_n}x_n) \tag{33}$$

for $0 \neq p \in \mathbb{R}[\bar{X}]$ and $x \in \mathbb{R}^n$, and

$$\operatorname{in}_w(pq) = \operatorname{in}_w(p)\operatorname{in}_w(q) \qquad (p, q \in \mathbb{R}[\bar{X}]) \tag{34}$$

Let $\mathcal{O}_w$ denote the valuation ring belonging to v_w and $\mathfrak{m}_w$ its maximal ideal. It is an easy exercise to show that a ring homomorphism $\lambda_w : \mathcal{O}_w \to \mathbb{R}(\bar{X})$ having kernel $\mathfrak{m}_w$ is defined by

$$\lambda_w\left(\frac{p}{q}\right) := \begin{cases} 0 & \text{if } v_w(p) > v_w(q) \\ \frac{\operatorname{in}_w(p)}{\operatorname{in}_w(q)} & \text{if } v_w(p) = v_w(q) \end{cases} \qquad (p, q \in \mathbb{R}[\bar{X}], q \neq 0), \tag{35}$$

i.e., λ_w is a place belonging to v_w.

We now give a concrete description of $S(T(g))$ using the notions just defined. This result is from Handelman [H1, Theorem III.3] and also included in [AT, Lemma 2.4]. For several reasons, we give here a third exposition of this proof. In contrast to [H1, III.2] and [AT, Lemma 2.3], we avoid the theory of polytopes and instead use some basic valuation theory and (inspired by [Bra, Lemma 1.10]) a fact from model theory. We believe that our viewpoint might be useful for the investigation of rings other than $A(g)$.

Theorem 18 (Handelman)**.** *For every $0 \neq g \in \mathbb{R}[\bar{X}]^+$ and $x \in S(T(g))$, there is some $w \in \mathbb{R}^n$ and $y \in (0,\infty)^n$ such that*

$$a(x) = \lambda_w(a)(y) \qquad \textit{for all } a \in A(g).$$

Proof. By Chevalley's Theorem [PD, A.1.10], we can extend the ring homomorphism $x : A \to \mathbb{R}$ to a place of $\mathbb{R}(\bar{X})$, i.e., we find a valuation ring $\mathcal{O} \supset A(g)$ of $\mathbb{R}(\bar{X})$ with maximal ideal $\mathfrak{m}$ and a ring homomorphism $\lambda : \mathcal{O} \to K$ into some extension field K of $\mathbb{R}$ with kernel $\mathfrak{m}$ such that $\lambda|_{A(g)} = x$. Let $v : \mathbb{R}(\bar{X}) \to \Gamma \cup \{\infty\}$ be a valuation belonging to $\mathcal{O}$ where Γ is (after extension) without loss of generality a nontrivial divisible ordered abelian group. Set

$$\Lambda := \{\alpha \in \operatorname{Log}(g) \mid v(\bar{X}^\alpha) = v(g)\} = \left\{\alpha \in \operatorname{Log}(g) \mid \lambda\left(\frac{\bar{X}^\alpha}{g}\right) \neq 0\right\}. \tag{36}$$

Now the first-order logic sentence

$$\exists u \exists v_1 \dots \exists v_n \left(\bigwedge_{\alpha \in \Lambda} \alpha_1 v_1 + \dots + \alpha_n v_n = u \wedge \bigwedge_{\alpha \in \operatorname{Log}(g) \setminus \Lambda} \alpha_1 v_1 + \dots + \alpha_n v_n > u \right)$$

in the language $\{+,<,0\}$ holds in Γ (take $v(g)$ for u and $v(X_i)$ for v_i). It is a well-known fact in basic model theory that all nontrivial divisible ordered abelian groups satisfy exactly the same first-order sentences in this language [Mar, Corollary 3.1.17]. In particular, the above sentence holds in $\mathbb{R}$, i.e., we find $w \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $\langle w, \alpha \rangle = c$ for all $\alpha \in \Lambda$ and $\langle w, \alpha \rangle > c$ for all $\alpha \in \operatorname{Log}(g) \setminus \Lambda$. It follows that $v_w(g) = -c$ and

$$\Lambda = \{\alpha \in \operatorname{Log}(g) \mid v_w(\bar{X}^\alpha) = v_w(g)\} = \left\{\alpha \in \operatorname{Log}(g) \mid \lambda_w\left(\frac{\bar{X}^\alpha}{g}\right) \neq 0\right\}. \tag{37}$$

In view of (36), (37) and (30), it remains only to show that there exists $y \in (0,\infty)^n$ such that

$$\lambda\left(\frac{\bar{X}^\alpha}{g}\right) = \lambda_w\left(\frac{\bar{X}^\alpha}{g}\right)(y) \qquad \text{for all } \alpha \in \Lambda. \tag{38}$$

Now set $m := \#\Lambda - 1 \in \mathbb{N}$ and write $\Lambda = \{\alpha^{(0)}, \dots, \alpha^{(m)}\}$. Assume for the moment that we have already shown the existence of some $y \in (0,\infty)^n$ satisfying

$$\lambda(\bar{X}^{\alpha^{(i)} - \alpha^{(0)}}) = y^{\alpha^{(i)} - \alpha^{(0)}} \qquad \text{for each } i \in \{1, \dots, m\}. \tag{39}$$

Then we get immediately that even

$$\lambda(\bar{X}^{\alpha^{(i)} - \alpha^{(j)}}) = y^{\alpha^{(i)} - \alpha^{(j)}} = \lambda_w(\bar{X}^{\alpha^{(i)} - \alpha^{(j)}})(y) \tag{40}$$

for $i,j \in \{0,\dots,m\}$. Writing $g = \sum_{\alpha\in\operatorname{Log}(g)} a_\alpha \bar{X}^\alpha$, we obtain

$$\begin{aligned}\lambda_w\left(\frac{g}{\bar{X}^{\alpha^{(i)}}}\right)(y)\lambda\left(\frac{\bar{X}^{\alpha^{(i)}}}{g}\right) &= \sum_{\alpha\in\operatorname{Log}(g)} a_\alpha\lambda_w\left(\frac{\bar{X}^{\alpha}}{\bar{X}^{\alpha^{(i)}}}\right)(y)\lambda\left(\frac{\bar{X}^{\alpha^{(i)}}}{g}\right)\\ \overset{(37)}{=} \sum_{j=0}^m a_{\alpha^{(j)}}\lambda_w\left(\frac{\bar{X}^{\alpha^{(j)}}}{\bar{X}^{\alpha^{(i)}}}\right)(y)\lambda\left(\frac{\bar{X}^{\alpha^{(i)}}}{g}\right) &\overset{(40)}{=} \sum_{j=0}^m a_{\alpha^{(j)}}\lambda\left(\frac{\bar{X}^{\alpha^{(j)}}}{\bar{X}^{\alpha^{(i)}}}\right)\lambda\left(\frac{\bar{X}^{\alpha^{(i)}}}{g}\right)\\ &= \sum_{j=0}^m a_{\alpha^{(j)}}\lambda\left(\frac{\bar{X}^{\alpha^{(j)}}}{g}\right) \overset{(36)}{=} \sum_{\alpha\in\operatorname{Log}(g)} a_\alpha\lambda\left(\frac{\bar{X}^{\alpha}}{g}\right) = \lambda\left(\frac{g}{g}\right) = \lambda(1) = 1\end{aligned}$$

which shows (38). Therefore we are left with showing that there is some $y \in (0,\infty)^n$ fulfilling (39). Set $\beta^{(i)} := \alpha^{(i)} - \alpha^{(0)} \in \mathbb{Z}^n$ and $z_i := \lambda(\bar{X}^{\beta^{(i)}})$ for $i \in \{1,\dots,m\}$. Note that for all $i \in \{1,\dots,m\}$,

$$z_i = \underbrace{\lambda\underbrace{\left(\frac{\bar{X}^{\alpha^{(i)}}}{g}\right)}_{\in T}}_{\neq 0 \text{ by } (36)}\underbrace{\lambda\underbrace{\left(\frac{\bar{X}^{\alpha^{(0)}}}{g}\right)}_{\in T}^{-1}}_{\neq 0 \text{ by } (36)} > 0$$

since $\varphi(T) \subseteq [0,\infty)$. Using

$$y^{\beta^{(i)}} = y_1^{\beta_1^{(i)}}\cdots y_n^{\beta_n^{(i)}} = e^{(\log y_1)\beta_1^{(i)}+\dots+(\log y_n)\beta_n^{(i)}},$$

taking logarithms in (39) and rewriting it in matrix form, we therefore have to show that there are $y_1',\dots,y_n' \in \mathbb{R}$ (corresponding to $\log y_1,\dots,\log y_n$) such that

$$\underbrace{\begin{pmatrix}\log z_1 & \dots & \log z_m\end{pmatrix}}_{=:L\in\mathbb{R}^{1\times m}} = \begin{pmatrix}y_1' & \dots & y_n'\end{pmatrix}\underbrace{\begin{pmatrix}\beta_1^{(1)} & \dots & \beta_1^{(m)}\\ \vdots & & \vdots\\ \beta_n^{(1)} & \dots & \beta_n^{(m)}\end{pmatrix}}_{=:B\in\mathbb{R}^{n\times m}} \tag{41}$$

Provided now that $\ker B \subset \ker L$, the mapping $\operatorname{im} B \to \mathbb{R} : Bv \mapsto Lv$ ($v \in \mathbb{R}^m$) is a well-defined linear map and can be linearly extended to a map $\mathbb{R}^n \to \mathbb{R}$ represented by a $1\times n$ matrix $\begin{pmatrix}y_1' & \dots & y_n'\end{pmatrix}$ satisfying (41).

Finally, we show $\ker B \subset \ker L$. Since all entries of B lie in the field $\mathbb{Q}$, $\ker B$ has a $\mathbb{Q}$-basis but then also $\mathbb{R}$-basis consisting of vectors $k \in \mathbb{Z}^m$. Therefore consider an arbitrary $k \in \mathbb{Z}^m$ with

$$\sum_{j=1}^m \beta_i^{(j)}k_j = 0 \qquad \text{for all } i \in \{1,\dots,m\}.$$

Taking the logarithm of

$$\begin{aligned}e^{(\log z_1)k_1+\dots+(\log z_m)k_m} &= z_1^{k_1}\cdots z_m^{k_m} = \lambda(\bar{X}^{\beta^{(1)}})^{k_1}\cdots\lambda(\bar{X}^{\beta^{(m)}})^{k_m}\\ &= \lambda(\bar{X}^{\beta^{(1)}k_1+\dots+\beta^{(m)}k_m}) = \lambda(\bar{X}^0) = \lambda(1) = 1 = e^0,\end{aligned}$$

we get indeed $k \in \ker L$. □

Corollary 19 (Handelman)**.** *For every $0 \neq g \in \mathbb{R}[\bar{X}]^+$ and $x \in S(T(g))$, there exist $w \in \mathbb{R}^n$ and $y \in (0,\infty)^n$ such that*

$$a(x) = \lim_{t\to\infty} a(e^{tw_1}y_1, \dots, e^{tw_n}y_n) \qquad \textit{for all } a \in A(g).$$

Proof. Rewrite the last theorem using (33) and (35). □

Proposition 20. *Suppose $f \in \mathbb{R}[\bar{X}]$ and let $l_1, l_2 \in \mathbb{N}$ be relatively prime in $\mathbb{Z}$. If it is true for f^{l_1} and f^{l_2} that all its sufficiently high powers have nonnegative coefficients, then the same is true for f.*

Proof. We may assume that *all* high powers of f^{l_1} and f^{l_2} have nonnegative coefficients (replace l_1 and l_2 for instance by sufficiently high powers of themselves). Now use a little exercise saying that, if $l_1, l_2 \in \mathbb{N}$ are relatively prime in $\mathbb{Z}$, the set $\mathbb{N}l_1 + \mathbb{N}l_2$ contains all sufficiently large integers. □

Lemma 21 (Handelman)**.** *Suppose $f \in \mathbb{R}[\bar{X}]$, $1 \leq l \in \mathbb{N}$ and $f^l \in \mathbb{R}[\bar{X}]^+$. Then there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and for all vertices α (i.e., extreme points) of* $\mathrm{New}(f)$*,*

$$(lk-1)\alpha + \mathrm{Log}(f) \subset \mathrm{Log}(f^{lk}).$$

Proof. It is convenient to work in the ring $\mathbb{R}[X_1, \dots, X_n, X_1^{-1}, \dots, X_n^{-1}] \subset \mathbb{R}(\bar{X})$ of Laurent polynomials. The Laurent monomials $\bar{X}^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ ($\alpha \in \mathbb{Z}^n$) form an $\mathbb{R}$-vector space basis of it. Extending the definitions in the obvious way, we can speak of $\mathrm{Log}(f) \subset \mathbb{Z}^n$ and $\mathrm{New}(f) \subset \mathbb{R}^n$ for any Laurent polynomial f. We now prove our claim even for Laurent polynomials f.

Since the polytope $\mathrm{New}(f)$ has only finitely many vertices, it suffices to show that the claimed inclusion of sets holds for a *fixed* vertex α and all large k. Replacing f by $\bar{X}^{-\alpha}f$, we can assume right away that $\alpha = 0$. Because the origin is now a vertex of $\mathrm{New}(f)$, we can choose $w \in \mathbb{R}^n$ such that $\langle w, \beta \rangle > 0$ for all $0 \neq \beta \in \mathrm{Log}(f)$. For all $0 \neq \beta, \gamma, \delta \in \mathrm{Log}(f)$ with $\beta = \gamma + \delta$, in the equality $\langle w, \beta \rangle = \langle w, \gamma \rangle + \langle w, \delta \rangle$ the two terms on the right hand side are then smaller than the left hand side. We need the following consequence from this: Calling a nonzero element of $\mathrm{Log}(f)$ an *atom* if it is not a sum of two nonzero elements of $\mathrm{Log}(f)$, every element of $\mathrm{Log}(f)$ can be written as a finite sum of atoms (the origin being the sum of zero atoms). Since $\mathrm{Log}(f)$ is finite, we can choose $k \in \mathbb{N}$ such that every element of $\mathrm{Log}(f)$ is a sum of at most k such atoms. On the other hand, because f^l has nonnegative coefficients, $\mathrm{Log}(f^{lk})$ consists of the sums of k elements of $\mathrm{Log}(f^l)$. Using $0 \in \mathrm{Log}(f)$, it is enough to show that all atoms are contained in $\mathrm{Log}(f^l)$. This is clear from the fact that an atom α can can be written as a sum of l elements from $\mathrm{Log}(f)$ only in a trivial way. In fact, the coefficient of $\bar{X}^\alpha$ in f^l is l times the coefficient of $\bar{X}^\alpha$ in f and therefore nonzero. □

Now we are prepared enough to give a proof of Handelman's result based on our membership criterion.

Theorem 22 (Handelman)**.** *Let $f \in \mathbb{R}[\bar{X}]$ be a polynomial such that f^k has no negative coefficients for some $k \geq 1$ and $f(1,1,\dots,1) > 0$. Then for all sufficiently large $k \in \mathbb{N}$, f^k has no negative coefficients.*

Proof. For any polynomial $p \in \mathbb{R}[\bar{X}]$, we write p^+ for the sum of its monomials with positive coefficients and p^- for the negated sum of its monomials with negative coefficients. So we always have $p = p^+ - p^-$, $p^+, p^- \in \mathbb{R}[\bar{X}]^+$ and

$\operatorname{Log}(p^+)\dot\cup\operatorname{Log}(p^-) = \operatorname{Log}(p)$. First, we prove the theorem under the additional assumption

$$\operatorname{in}_w(f) \in \mathbb{R}[\bar{X}]^+ \qquad \text{for all } w \in \mathbb{R}^n \text{ with } \operatorname{in}_w(f) \neq f. \tag{42}$$

By Lemma 21, we can choose $k \in \mathbb{N}$ such that $g := f^k$ has no negative coefficients and

$$(k-1)\alpha + \operatorname{Log}(f) \subset \operatorname{Log}(g) \qquad \text{for all vertices } \alpha \text{ of } \operatorname{New}(f). \tag{43}$$

Pick an arbitrary vertex α_0 of $\operatorname{New}(f)$. Then we have for all $N \in \mathbb{N}$,

$$a := \frac{\bar{X}^{(k-1)\alpha_0} f}{g} = (1 - Nc_1)\frac{\bar{X}^{(k-1)\alpha_0} f^+}{g} + (Nc_2 - 1)\frac{\bar{X}^{(k-1)\alpha_0} f^-}{g} \tag{44}$$

where

$$c_1 := \sum_\alpha \frac{\bar{X}^{(k-1)\alpha} f^-}{g}, \qquad c_2 := \sum_\alpha \frac{\bar{X}^{(k-1)\alpha} f^+}{g}$$

and the indices of summation run over all vertices α of $\operatorname{New}(f)$. We will show that for N sufficiently big, (44) serves as an identity like it is required in Theorem 12 which we are going to apply to the ring $A := A(g)$ together with its archimedean semiring $T := T(g)$. To do this, first of all, observe that all fractions appearing in (44) lie in A by (43).

Claim 1: $a > 0$ on $(0, \infty)^n$. From $f^k = g \in \mathbb{R}[\bar{X}]^+$, it follows that $a^k > 0$ on $(0,\infty)^n$. Using the continuity of a on the connected space $(0,\infty)^n$, we obtain either $a > 0$ on $(0,\infty)^n$ or $a < 0$ on $(0,\infty)^n$. The latter can be excluded using the hypothesis $f(1, 1, \ldots, 1) > 0$

Claim 2: $a \geq 0$ on $S(T)$. This follows from Claim 1 and Corollary 19.

Claim 3: $c_1 = 0$ on $S(T) \cap \{a = 0\}$. Let $w \in \mathbb{R}^n$. According to Theorem 18, we would have to show that $\lambda_w(a)(y) = 0$ implies $\lambda_w(c_1)(y) = 0$ for all $y \in (0,\infty)^n$. In fact, we show that $\lambda_w(c_1) \neq 0$ implies $\lambda_w(a) = a$ which is clearly more by Claim 1. So suppose that $\lambda_w(c_1) \neq 0$. Then there is some vertex α of $\operatorname{New}(f)$ with $v_w(g) = v_w(\bar{X}^{(k-1)\alpha} f^-)$. Using $v_w(g) = kv_w(f)$, $v_w(\bar{X}^\alpha) \geq v_w(f)$ and $v_w(f^-) \geq v_w(f)$, we get $v_w(f) = v_w(f^-)$ whence $\operatorname{in}_w(f) \notin \mathbb{R}[\bar{X}]^+$. From (42), we now deduce $\operatorname{in}_w(f) = f$. This means that for all exponent tuples $\beta \in \mathbb{N}^n$ appearing in f, $\langle w, \beta\rangle = -v_w(f)$ is constant. Being vertices of $\operatorname{New}(f)$, both α_0 and α are among these β. We obtain therefore $v_w(\bar{X}^{(k-1)\alpha_0} f) = (k-1)v_w(\bar{X}^{\alpha_0}) + v_w(f) = kv_w(f) = v_w(f^k) = v_w(g)$. Exploiting the definition (35) of λ_w together with $\operatorname{in}_w(\bar{X}^{(k-1)\alpha_0} f) = \bar{X}^{(k-1)\alpha_0}\operatorname{in}_w(f) = \bar{X}^{(k-1)\alpha_0} f$ and $\operatorname{in}_w(g) = \operatorname{in}_w(f^k) = \operatorname{in}_w(f)^k = f^k = g$, we see that $\lambda_w(a) = a$.

Claim 4: $\operatorname{New}(f) = \operatorname{New}(f^+)$. Of course, we have $\operatorname{New}(f) \supset \operatorname{New}(f^+)$ since $\operatorname{Log}(f) \supset \operatorname{Log}(f^+)$. For the other inclusion, it clearly suffices to show that every vertex α of $\operatorname{New}(f)$, is contained in $\operatorname{Log}(f^+)$. But for such a vertex α, $\operatorname{in}_w(f) = \{\lambda\bar{X}^\alpha\}$ for some $\lambda \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$. Except in the case where $f = \lambda\bar{X}^\alpha$, it follows from (42) that $\lambda > 0$ whence $\alpha \in \operatorname{Log}(f^+)$. If $f = \lambda\bar{X}^\alpha$, then $\lambda > 0$ follows from $f(1, 1, \ldots, 1) > 0$.

Claim 5: $c_2 > 0$ on $S(T)$. Let $w \in \mathbb{R}^n$. By Theorem 18, $\lambda_w(c_2)(y) > 0$ for all $y \in (0,\infty)^n$ is what we would have to show. By definition of λ_w it is enough to show that $\lambda_w(c_2) \neq 0$ since $\bar{X}^{(k-1)\alpha} f^+$ has no negative coefficients. We obtain from Claim 4 that $v_w(f^+) = v_w(f)$. Choose a vertex α of $\operatorname{New}(f)$ such that $v_w(f) = v_w(\bar{X}^\alpha)$. Then $v_w(\bar{X}^{(k-1)\alpha} f^+) = (k-1)v_w(\bar{X}^\alpha) + v_w(f^+) = kv_w(f) = v_w(f^k) = v_w(g)$. Therefore $\lambda_w(c_2) \neq 0$ as desired.

Regarded as a continuous real-valued function on the compact space $S(T)$, c_2 is bounded from below by some positive real number by Claim 5. Consequently, we can choose $N \in \mathbb{N}$ so large that $Nc_2 - 1 > 0$ on the whole of $S(T)$, in particular on $S(T) \cap \{a = 0\}$. By Claim 3, we have that $1 - Nc_1 = 1 > 0$ on $S(T) \cap \{a = 0\}$. Altogether, we can apply Theorem 12 and see that $a \in T$. By definition of $T = T(g)$, this means that $g^m \bar{X}^{(k-1)\alpha_0} f \in \mathbb{R}[\bar{X}]^+$ for some $m \in \mathbb{N}$. Omitting $\bar{X}^{(k-1)\alpha_0}$ does not change this fact, so that $f^{km+1} = g^m f \in \mathbb{R}[\bar{X}]^+$. At the same time, of course, $f^{km} = g^m \in \mathbb{R}[\bar{X}]^+$. Proposition 20 yields now that all sufficiently high powers of f lie in $\mathbb{R}[\bar{X}]^+$.

Thus we have shown the theorem under the assumption (42). Now in the general case, we proceed by induction on the number of monomials appearing in f. The case where f has only one monomial is trivial. Now suppose that f has at least two monomials. The hypothesis implies clearly that

$$f > 0 \qquad \text{on } (0, \infty)^n. \tag{45}$$

Let $w \in \mathbb{R}^n$ such that $\mathrm{in}_w(f)$ has less monomials than f. For some $k \geq 1$, $(\mathrm{in}_w(f))^k = \mathrm{in}_w(f^k) \in \mathbb{R}[\bar{X}]^+$ by the hypotheses on f. Evaluating this at $(1, 1, \ldots, 1)$, we see that $\mathrm{in}_w(f)$ does not vanish at this point. Moreover, it is non-negative at the same point by (33) and (45). Altogether, we can apply the induction hypothesis on $\mathrm{in}_w(f)$ to get that $\mathrm{in}_w(f^k) = (\mathrm{in}_w(f))^k \in \mathbb{R}[\bar{X}]^+$ for all large k.

Since $\{\mathrm{in}_w(f) \mid w \in \mathbb{R}^n\}$ is of course finite, this shows that we find $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ and $w \in \mathbb{R}^n$ with $\mathrm{in}_w(f) \neq f$, $\mathrm{in}_w(f^k) \in \mathbb{R}[\bar{X}]^+$. This shows that (42) is satisfied with f replaced by f^k for any $k \geq k_0$ (note that $\mathrm{in}_w(f^k) \neq f^k$ implies trivially $\mathrm{in}_w(f) \neq f$). In particular, we find $l_1, l_2 \in \mathbb{N}$ that are relatively prime in $\mathbb{Z}$ such that (42) holds with f replaced by f^{l_1} and f^{l_2}, e.g., take $l_1 := k_0$ and $l_2 := k_0 + 1$. By the special case of the theorem already proved, we get that $(f^{l_1})^k$ and $(f^{l_2})^k$ have no negative coefficients for all large k. According to Proposition 20, this means that all sufficiently high powers of f have only nonnegative coefficients. □

Corollary 23 (Handelman)**.** *If some odd power of a real polynomial in several variables has only nonnegative coefficients, then so do all sufficiently high powers.*

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Universität Konstanz, Fachbereich Mathematik und Statistik, 78457 Konstanz, Allemagne

E-mail address: `Markus.Schweighofer@uni-konstanz.de`