Thorsten Theobald: "Real Algebraic Geometry and Optimization". AMS, 2024, xv + 293 pp

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In memory of Prof. Dr. Alexander Prestel[1]
(* 1941 in Berchtesgarden, † 2024 in Konstanz) [2]

The book under review is a textbook that serves as an introduction to the basics of *real algebraic geometry*, to the framework and instances of *conic optimization* and to a field that grew since the turn of the millennium out of the interplay of both, namely *polynomial optimization*. The seventh of the ten chapters of the book is exactly about this latter field but all the other chapters are directly or indirectly related to it.

Hence we start by explaining what is a *polynomial optimization problem* (POP). It is the problem of minimizing (or maximizing) a polynomial objective $p(x)$ over all $x \in \mathbb{R}^n$ subject to a system of polynomial inequalities

$$g_1(x) \geq 0, \ldots, g_m(x) \geq 0.$$

The feasible sets of POPs, i.e., the solution sets of such systems are called *basic closed semialgebraic sets*. A general *semialgebraic set* is a boolean combination of basic closed ones. Solving such POPs is very hard in general, even in the unconstrained case $m = 0$ and already when p is of degree four, as the author explains on the last pages of his appendix.

On the other hand, there is a general procedure called *real quantifier elimination* which solves this and even many more general problems in theory algorithmically. For the above POP, the first step would be to eliminate the n existential quantifiers in the formula

$$\exists x_1 \ldots \exists x_n (y = p(x) \wedge g_1(x) \geq 0 \wedge \ldots \wedge g_m(x) \geq 0)$$

interpreted in the reals. In general, as Tarski knew already in the 1930s, one can eliminate all quantifiers over the reals in formulas that are built up from polynomial inequalities inductively by the quantifiers $\forall, \exists$ and the logical connectives $\vee$, $\wedge$ and $\neg$. Geometrically, this means that projections of semialgebraic sets are again semialgebraic. It will later become important that the same elimination procedure works not only over the reals but even simultaneously for each so called *real closed field*. This is essentially *Tarski's transfer principle* we will mention later. Real closed fields go back to Artin and Schreier in the 1920s and generalize the field of real numbers in very much the same way than algebraically closed fields generalize the field of complex numbers. All this is treated in Chapter 3 of the reviewed book. This quantifier elimination procedure relies on classical methods of real root counting of univariate polynomials that are treated in Chapter 1.

While of theoretical importance, Tarski's procedure is even on small examples too inefficient. In the 1970s, Collins discovered a new method for real quantifier elimination

[1] http://www.math.uni-konstanz.de/~schweigh/24/ObituaryPrestel

[2] This preprint has not undergone peer review or any post-submission improvements or corrections. The Version of Record of this article is published in *Jahresbericht der Deutschen Mathematiker-Vereinigung* (2024, 126:313-321), and is freely available online at https://doi.org/10.1365/s13291-024-00288-0.

that can deal in practice with more realistic albeit still small instances. It is based on *cylindrical algebraic decomposition*. In the above example of a POP, it would first decompose the set

$$\{(x,y) \in \mathbb{R}^n \times \mathbb{R} \mid p(x) - y = 0, g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$$

into finitely many connected semialgebraic sets on which each of the $m+1$ defining polynomials has constant sign. Omitting some of the technical details (such as computing with real algebraic numbers), the author presents this in Chapter 4.

The question persists of how to *practically* solve a POP, at least approximately, at least in certain cases. The easiest case where even large instances can be solved effectively in theory and in practice, is the case where the objective p and the constraining polynomials $g_1, \ldots, g_m$ all have degree at most 1 (i.e., they are linear as a polynomial or affine-linear as a function). In this case, the POP becomes simply a *linear program* (LP) and its feasible set is a *polyhedron*. An LP is the easiest case of a *conic optimization problem* which is the problem of minimizing a linear objective function over an affine slice of a closed convex cone. In the case of an LP, the cone is just the nonnegative orthant $[0,\infty)^n$.

A strange but useful way of viewing the nonnegative orthant, is to see it as the cone of *positive semidefinite* (psd) real diagonal matrices. Now all real symmetric (psd) matrices are orthogonally conjugate to a real diagonal (psd) matrix. Taking as a cone the cone of all psd matrices instead of the nonnegative orthant, the corresponding conic optimization problem gets a *semidefinite program* (SDP). The feasible sets of SDPs are called *spectrahedra*. Spectrahedra form a very interesting compromise between polyhedra and arbitrary closed convex semialgebraic sets, as they still behave in many respects like polyhedra but they allow for some roundness in their shapes. This is the message of Chapter 2 of the book under review. Chapter 5 introduces the reader to linear, semidefinite and conic optimization, in this order of increasing generality, and its duality theory. At the end of that chapter, the author gives an idea of how to solve these optimization problems using *interior point algorithms* driven by *self-concordant barrier functions* whose theory has been developed by Nesterov and Nemirovski in the 1990s. For example, the negated logarithm of the determinant is an easily computable such barrier function on the cone of psd matrices and this is why SDPs can mostly still be solved efficiently (although not as fast as LPs).

If our POP is a *quadratic optimization* problem (i.e., the polynomials p and g_i are of degree at most 2), then it is in general already very hard to solve. In fact, many NP-hard combinatorial problems can easily be encoded as a quadratic optimization problem. Even LPs augmented by the 0-1-constraints $x_i^2 = x_i$ (which could be written as $x_i^2 \geq x_i$ and $x_i^2 \leq x_i$ to fit into the framework above) are usually intractable. In the 1990s it became popular in the field of *combinatorial optimization* to *relax* such a quadratic optimization problem in the following way: In a first step, one replaces all occurrences of a quadratic monomial x_ix_j with $i \leq j$ by a new variable x_{ij}. This extremely naïf procedure would usually result in an LP that approximates the original quadratic POP very poorly. To improve the quality, one turns this LP into an SDP by adding the constraint that the matrix $X := (x_{ij})_{i,j\in\{0,\ldots,n\}}$ is psd where $x_{00} := 1$, $x_{0i} := x_i$ for $i > 0$ and $x_{ji} := x_{ij}$ for $i > j$. This additional semidefinite constraint memorizes *partially* where the new variables x_{ij} came from. The *full* memory would need in addition that X is a rank one matrix. This condition is sacrificed for tractability. The price to pay is that the SDP only *approximates* the original POP. How good the approximation is, depends in a very subtle way on the original POP and is usually analyzed through an every-growing zoo of (often very subtle) *rounding procedures* that try to round an optimal solution of the SDP to an approximately optimal solution of the POP. This established semidefinite programming as a major tool in the area of *approximation algorithms* for combinatorial problems [4] which is beyond the scope of the book under review.

Pioneered by Lasserre, Parrilo and others, it became apparent around the year 2000, that some of these ideas elegantly generalize to POPs of higher degree: One now introduces new variables for all monomials of a fixed *relaxation degree* d which can be chosen higher than the degree of the POP. Without going into details, one gets now an (for big d unfortunately huge) SDP with several semidefinite constraints: The analog to the matrix X from a above now has a multivariate Hankel structure and corresponds to a linear form on the space of polynomials of degree d. Requiring it to be psd amounts to say that this linear form maps *sum-of-squares polynomials* into the nonnegative reals. In addition, each constraint $g_i(x) \geq 0$ of the original POP, now gives rise to an additional semidefinite constraint meaning that this same linear form maps g_i *multiplied* by a sum-of-squares polynomial into the nonnegative reals. In this way, one can approximate the original POP by an SDP depending on the relaxation degree d. When d increases, the approximation gets better or at least not worse. *In practice,* one chooses d small enough so that one can solve the corresponding SDP in a small amount of time. *In theory,* most results treated in the book under review are however about the limit $d \to \infty$. For example, using *Putinar's Positivstellensatz* explained below, one can show that the optimal value of the SDP approaches the optimal value of the original POP as the relaxation degree d goes to infinity under the so-called *Archimedean condition* on the $g_1, \dots, g_m$. This gives at least hope that in certain cases, even for a d that still allows to solve the SDP in practice, the limit is attained and optimal points of the original POP can be extracted from an optimal solution of the SDP. Some criteria for when this works are based on the *flat extension theorem* of Curto and Fialkow which the author decided to present without proof. This is a criterion for when a linear form as mentioned above (mapping weighted sum-of-squares polynomials to the nonnegative numbers) is integration with respect to a measure. It is thus a solution to the *truncated moment problem*, namely the problem when a finite real (multi-)sequence is the sequence of moments of a measure. The author also emphasizes the role of the (non-truncated) *moment problem* in this regard. These beautiful ideas are treated in Chapter 7 of the book under review.

Concerning still the just reviewed Chapter 7, three more comments of different nature are in order: First, at the beginning of the chapter, the author introduces the reader smoothly to the topic of polynomial optimization by first presenting a variant using LP relaxations instead of SDP relaxations in the case where the feasible set of the POP is a polytope. Similar variants would in fact also work for arbitrary basic closed semialgebraic feasible sets. But LP relaxations have a couple of drawbacks. They exhibit often slow convergence in practice for $d \to \infty$, often without hope for finite convergence or for extraction of optimal solutions for the POP.

Second, not only in the reviewed book but also in the whole existing literature there is so far a lack of theoretical results, relating the POP and its SDP relaxations for *fixed* degree d of moderate size. There is one exception to this: In the combinatorial framework with, say, 0-1-constraints $x_i^2 = x_i$ as mentioned above, but this time with $d > 2$, say $d = 4$ or d about $\log_2 n$ where n is the number of variables x_i in the original POP, people working in the area of *approximation algorithms* have again done a great deal of work. This is again based on all kinds of tricky rounding procedures and has had a huge impact on modern computational complexity theory in theoretical computer science. For good reasons, this is not addressed in the book under review. Recent treaties introducing to this subject are [3, 6].

Third, Chapter 7 also links the theory of SDP relaxations of POPs to one of the most classical threads of research in real algebraic geometry, namely the theory of sum-of-squares representations of polynomials. Indeed, it turns out that the *dual* SDP (to the SDP relaxation of the POP of degree d) is the problem of maximizing a certifiable lower bound $c \in \mathbb{R}$ for the polynomial objective p on the feasible set S of the POP (which was defined by the system of polynomial inequalities $g_1(x) \geq 0, \ldots, g_m(x) \geq 0$). Letting g_0 denote the constant 1 polynomial, the corresponding certificate is a weighted sum-of-squares certificate

$$p - c = \sum_{i=0}^{m} s_i g_i$$

with sum-of-squares polynomials s_i of degree small enough such that the degrees of the $s_i g_i$ do not exceed d. It is *Putinar's Positivstellensatz* that certifies the existence of this representation under the *Archimedean condition* on the $g_1, \ldots, g_m$ and for each *strict* lower bound c of p on S. Hereby, the degree bound d however often has to approach infinity while the lower bound c approaches the infimum of p on S. This means that many monomials appearing in the individual terms $s_i g_i$ of the sum on the right hand side have to cancel out after summing over $i \in \{0, \ldots, m\}$. From classical computational algebraic geometry and the theory of Gröbner bases, this phenomenon of monomial cancellation frequently occurs when the multipliers are just assumed to be arbitrary polynomials rather than sum-of-squares polynomials. In real algebraic geometry, the Archimedean condition is a technical condition that allows for these cancellations even though the s_i are sum-of-squares polynomials. This condition implies that S is compact. Conversely, if S is compact, the Archimedean condition can be ensured by adding a "big ball constraint" to the g_i without changing S.

The just outlined duality is the reason why the theory of sum-of-squares representations of polynomials is the theoretical underpinning of modern polynomial optimization. The author thus develops this classical part of real algebraic geometry already in the preceding chapter which is Chapter 6. Here finally general real closed fields and Tarski's transfer principle from Chapter 3 come into play. *Hilbert's Nullstellensatz* from the 1890s is a classical result from algebraic geometry that says (in its most basic form from which more elaborate forms can easily be derived) that a system of polynomial inequalities $h_1(x) = 0, \ldots, h_m(x) = 0$ over an *algebraically closed* field C (such as $\mathbb{C}$) has no solution in $x \in C^n$ (if and) only if

$$1 = \sum_{i=1}^{m} f_i h_i$$

for some polynomials f_i. The *Positivstellensatz* from real algebraic geometry (due to Krivine in the 1960s and rediscovered by Prestel and Stengle in the 1970s) says (again in its most basic form from which again more elaborate forms can easily be derived) that a system of polynomial inequalities $g_1(x) \geq 0, \dots, g_m(x) \geq 0$ over a *real closed* field R (such as $\mathbb{R}$) has no solution in $x \in R^n$ (if and) only if

$$-1 = \sum_{\alpha \in \{0,1\}^m} s_\alpha g_1^{\alpha_1} \cdots g_m^{\alpha_m}$$

for some sum-of-squares polynomials s_α. The proof is non-constructive and goes by contraposition:

Suppose -1 cannot be written in this way. Then enlarge the system of inequalities by Zorn's lemma (allowing it now to be infinite) as much as possible so that -1 does still not have a corresponding representation. From this somehow formally consistent infinite system of inequalities, one constructs now a real closed extension field R' of R where this enlarged infinite system has a solution. In particular, the original finite system of inequalities has a solution over R'. By Tarski's transfer it does so over R.

The Positivstellensatz thus characterizes when a POP is infeasible but is for several reasons not yet well adapted to approximately solve POPs. The construction of R' in its proof needs basically the notion of the real spectrum which is implicitly developed in the same Chapter 6 exactly to the point that is needed (not explicitly defined, not on arbitrary commutative rings, without its topology). These arguments get a little less technical for a precursor of the Positivstellensatz which is the solution that Artin obtained in the 1920s to Hilbert's 17th problem. Artin's theorem is thus presented before the Positivstellensatz although it can easily be derived from it. Better suited for POPs is *Putinar's Positivstellensatz* already mentioned above which, although proved only in the 1990s, neither needs real closed fields nor the real spectrum for its proof. Another result presented in Chapter 6 is the celebrated *Schmüdgen's Positivstellensatz* from the 1990s. Its proof is based on the Positivstellensatz and thus it is much deeper than Putinar's result. It is here (and usually elsewhere) presented as a variant of Putinar's theorem but really its added value compared to Putinar's theorem is that it characterizes compactness of the feasible set of a POP in a similar way than the Positivstellensatz characterizes its emptiness (replacing -1 on the left hand side of the corresponding algebraic identity by $r^2 - \sum_i x_i^2$ for some $r \in \mathbb{R}$). Finally, other theorems playing a similar role than Putinar's theorem but for LP instead of SDP relaxations are also presented.

Whereas in the previous chapters, spectrahedra appeared as a tool, they become the object of investigation in Chapter 9. It is astonishing that all kinds of questions that are clarified for polyhedra are widely open for spectrahedra. How to decide efficiently emptiness, boundedness and containment of spectrahedra? What is a geometric characterization of spectrahedra? What are the projections of spectrahedra? The author discusses these questions and provides partial answers. The proofs of the easier statements are given and are omitted for the harder ones. A very active research question is whether spectrahedra are (up to technicalities) precisely the so-called *rigidly convex* sets. This has been conjectured by Helton and Vinnikov in 2007 who also have proven it in the plane. Two years later, Helton and Nie conjectured that the projections of spectrahedra are exactly the convex semi-algebraic sets but this has been disproved by Scheiderer in 2018.

Chapter 9 is a concise first introduction to the beautiful topic of univariate and multivariate stable and hyperbolic polynomials. Very roughly speaking, these are polynomials with many real zeros which have their place in combinatorics and optimization, respectively, and only in the last decades have regained popularity amongst real algebraic geometers. As an example of their importance in combinatorics, the author defines the matching polynomial of a graph and uses it to show that the number of matchings of a graph of a certain size first increases and then decreases with growing size. As an example of their role in conic optimization, the author defines *hyperbolicity cones* whose affine sections are exactly the already mentioned rigidly convex sets. These have nice self-concordant barrier functions that enable interior point methods to navigate over them. This leads to a whole subbranch of conic optimization called *hyperbolic optimization*. Hyperbolic programming over the cone of psd matrices is just semidefinite programming. The applications of hyperbolic optimization beyond semidefinite programming are however still rare and, even more, it is not clear whether each hyperbolicity cone is a linear section of the cone of psd matrices. This latter conjecture is equivalent to the conjecture of Helton and Vinnikov from 2007 already mentioned. It is related to the topic of determinantal representations of hyperbolic polynomials. With respect to this latter topic, the author states the Helton-Vinnikov theorem from 2007 that says that a hyperbolic polynomial in three variables is sort of a "joint characteristic polynomial" of a pair of symmetric matrices. All known proofs of this theorem are very deep and consequently this statement is given without proof. The author presents also many classical results related to *interlacing* of real zero sets of polynomials. The beautiful and important theory of stability preservers [14, 19] is however not approached.

Finally, in Chapter 10, the author presents two other instances of tractable conic optimization that have proved to be very useful for polynomial (and even *signomial*) optimization. The corresponding cones are the *exponential cone* and the *relative entropy cone* which are essentially reparametrizations of each other. Mainly in the last ten years, these cones have been linked by many researchers, including Theobald himself, to new non-negativity certificates that not only go beyond the traditional sum-of-squares representation but even seem to lie in a sense orthogonal to them. Notably, these new certificates are mainly based on the inequality of arithmetic and geometric means which is particularly resistant to the sum-of-squares method.

In conclusion, the book under review provides an excellent introduction to the fascinating interplay between real algebraic geometry and optimization that has emerged since the turn of the millennium. It is primarily aimed at master's students but is also suitable for very good bachelor's students and even doctoral candidates if they still need to get acquainted with the area. Although it is a classical textbook that offers in principle very detailed explanations and only occasionally omits proofs, it can in no way cover the topics exhaustively. Nevertheless, it has the potential to become a standard reference for research articles. Among the existing textbooks, we consider it to be the one that is best balanced between real algebraic geometry and optimization. Very pedagogically, the author often chooses accessibility over the utmost generality and efficiency, and allows redundancy when it helps the reader to understand the ideas.

Since a review should not only consist of praise, we mention some points of criticism although they are minor: Sometimes, no (or at least no direct or precise) references are given for unproven statements, not even in the notes that conclude each chapter. This occurs particularly often when complexity bounds are mentioned, for example on Pages 76, 110 and 137. In certain places, one gets the feeling that something is being presented as straightforward when, upon reflection, it is actually not so easy to prove, such as the uniqueness of the minimal degree defining polynomial for a given algebraic interior on Page 178 or the fact that every psd form can be written as a finite sum of extremal psd forms on Page 106, attributed by the author to Carathéodory but likely going back to Minkowski. In several instances, a more algebraic perspective would have made things clearer. For example, Theorem 7.35 on Page 164 follows (even with squares instead of *sums of* squares) immediately from the fact that every reduced finite-dimensional real commutative algebra is (by the Chinese remainder theorem) isomorphic to a direct product of finitely many copies of $\mathbb{R}$ and $\mathbb{C}$.

We conclude by what we think are the most important books, lecture notes and survey articles that compete against or complete Theobald's very appealing book. The standard resource for more classical parts of real algebraic geometry like, for example, the topology of real algebraic sets, is still the book by Bochnak, Coste and Roy [2]. For symbolic computation in real algebraic geometry that goes way beyond cylindrical algebraic decomposition, the most important resource is the book by Basu, Pollack and Roy [1]. The best references for using SDPs to solve POPs more from the perspective of combinatorial optimization and theoretical computer science are perhaps the book by Gärtner and Matousek [4] that covers the earlier development, the book of Fleming, Kothari and Pitassi [3] that ties the method also to statistics and proof systems and the wonderful (unpublished) lecture notes of Kunisky [6] with connections to the theories of random graphs and tensors. These last resources aim mainly at designing theoretical polynomial time (approximation) approximations which might not run well in practice. To have a chance of solving larger POPs by SDPs *practically*, one actually has usually to do a lot of problem-specific adaptions of the method such as exploiting sparsity in the sum-of-squares method. Books that take that into account and at the same time make the sum-of-squares method work for problems that go beyond polynomial optimization (such as solving or controlling differential equations) are the books by Lasserre [7, 8], by Henrion and Korda [5] and, very recently, by Magron and Wang [10]. The compact survey of Laurent and its unpublished update [9] is an excellent alternative for those who are primarily interested in solving POPs. The recent longer book by Nie [13] is always worth to consult for specific problems and applications. The books by Powers [15], by Prestel and Delzell [16], by Marshall [11], the lecture notes by the author of this review [18] (entirely available as videos) and the brand new book by Scheiderer [17] are introductions to real algebraic geometry with a strong focus on positivity and sums of squares. Here, [15] is particularly accessible for newcomers, [16] contains deep material on the subtleties of the Archimedean condition and [11, 18, 17] provide important variants of Putinar's Positivstellensatz for *nonnegative* (instead of *positive*) polynomials that are particularly interesting for polynomial optimization. The book by Netzer and Plaumann [12] is an interesting alternative to these sources that focuses more on hyperbolic polynomials. The surveys of Pemantle [14] and Wagner [19] are finally excellent roundups on hyperbolic and stable polynomials and their applications.

Along with the mentioned and numerous other sources, Theobald's book will allow more mathematicians to experience the exciting and ongoing development in between real algebraic geometry and optimization that began a quarter century ago.

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